

# THE LAGRANGIAN AND HAMILTONIAN ASPECTS OF THE ELECTRODYNAMIC VACUUM-FIELD THEORY MODELS

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**ABSTRACT.** We review the modern classical electrodynamics problems and present the related main fundamental principles characterizing the electrodynamical vacuum-field structure. We analyze the models of the vacuum field medium and charged point particle dynamics using the developed field theory concepts. There is also described a new approach to the classical Maxwell theory based on the derived and newly interpreted basic equations making use of the vacuum field theory approach. In particular, there are obtained the main classical special relativity theory relations and their new explanations. The well known Feynman approach to Maxwell electromagnetic equations and the Lorentz type force derivation is also discussed in detail. A related charged point particle dynamics and a hadronic string model analysis is also presented. We also revisited and reanalyzed the classical Lorentz force expression in arbitrary non-inertial reference frames and present some new interpretations of the relations between special relativity theory and its quantum mechanical aspects. Some results related with the charge particle radiation problem and the magnetic potential topological aspects are discussed. The electromagnetic Dirac-Fock-Podolsky problem of the Maxwell and Yang-Mills type dynamical systems is analyzed within the classical Dirac-Marsden-Weinstein symplectic reduction theory. Based on the Gelfand-Vilenkin representation theory of infinite dimensional groups and the Goldin-Menikoff-Sharp theory of generating Bogolubov type functionals the problem of constructing Fock type representations and retrieving their creation-annihilation operator structure is analyzed. An application of the suitable current algebra representation to describing the non-relativistic Aharonov-Bohm paradox is presented. The current algebra coherent functional representations are constructed and their importance subject to the linearization problem of nonlinear dynamical systems in Hilbert spaces is demonstrated.

## 1. CLASSICAL RELATIVISTIC ELECTRODYNAMICS MODELS REVISITING: LAGRANGIAN AND HAMILTONIAN ANALYSIS

**1.1. Introductory setting.** Classical electrodynamics is nowadays considered [97, 116] as the most fundamental physical theory, largely owing to the depth of its theoretical foundations and wealth of experimental verifications. In the work we describe a new approach to the classical Maxwell theory, based on a vacuum field medium model, and reanalyze some of the modern classical electrodynamics problems related with the description of a charged point particle dynamics under external electromagnetic field. We remark here that under "*a charged point particle*" we as usually understand an elementary material charged particle whose internal spatial structure is assumed to be unimportant and is not taken into account, if the contrary is not specified.

The important physical principles, characterizing the related electrodynamical vacuum field structure and based on the least action principle, we discuss subject to different charged point particle dynamics, based on the fundamental least action principle. In particular, the main classical relativistic relationships, characterizing the charge point particle dynamics, we obtain by means of the least action principle within the Feynman's approach to the Maxwell electromagnetic equations and the Lorentz type force derivation. Moreover, for each least action principle constructed in the work, we describe the corresponding Hamiltonian pictures and present the related energy conservation laws. Making use of the developed modified least action approach a classical hadronic string model is analyzed in detail.

As the classical Lorentz force expression with respect to an arbitrary inertial reference frame is related with many theoretical and experimental controversies, such as the relativistic potential energy impact into the charged point particle mass, the Aharonov-Bohm effect [2] and the

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Abraham-Lorentz-Dirac radiation force [82, 39, 97] expression, the analysis of its structure subject to the assumed vacuum field medium structure is a very interesting and important problem, which was discussed by many physicists including E. Fermi, G. Schott, R. Feynman, F. Dyson [55, 141, 57, 49, 50, 63] and many others. To describe the essence of the electrodynamic problems related with the description of a charged point particle dynamics under external electromagnetic field, let us begin with analyzing the classical Lorentz force expression

$$(1.1) \quad dp/dt = F_\xi := \xi E + \xi u \times B,$$

where  $\xi \in \mathbb{R}$  is a particle electric charge,  $u \in T(\mathbb{R}^3)$  is its velocity [1, 20] vector, expressed here in the light speed  $c$  units,

$$(1.2) \quad E := -\partial A/\partial t - \nabla\varphi$$

is the corresponding external electric field and

$$(1.3) \quad B := \nabla \times A$$

is the corresponding external magnetic field, acting on the charged particle, expressed in terms of suitable vector  $A : M^4 \rightarrow \mathbb{E}^3$  and scalar  $\varphi : M^4 \rightarrow \mathbb{R}$  potentials. Here " $\nabla$ " is the standard gradient operator with respect to the spatial variable  $r \in \mathbb{E}^3$ , " $\times$ " is the usual vector product in three-dimensional Euclidean vector space  $\mathbb{E}^3 := (\mathbb{R}^3, \langle \cdot, \cdot \rangle)$ , which is naturally endowed with the classical scalar product  $\langle \cdot, \cdot \rangle$ . These potentials are defined on the Minkowski space  $M^4 \simeq \mathbb{R} \times \mathbb{E}^3$ , which models a chosen laboratory reference frame  $\mathcal{K}$ . Now, it is a well known fact [97, 116, 57, 148] that the force expression (1.1) does not take into account the dual influence of the charged particle on the electromagnetic field and should be considered valid only if the particle charge  $\xi \rightarrow 0$ . This also means that expression (1.1) cannot be used for studying the interaction between two different moving charged point particles, as was pedagogically demonstrated in classical manuals [97, 57]. As the classical Lorentz force expression (1.1) is a natural consequence of the interaction of a charged point particle with an ambient electromagnetic field, its corresponding derivation based on the general principles of dynamics, was deeply analyzed by R. Feynman and F. Dyson [57, 49, 50].

Taking this into account, it is natural to reanalyze this problem from the classical, taking only into account the Maxwell-Faraday wave theory aspect, specifying the corresponding vacuum field medium. Other questionable inferences from the classical electrodynamics theory, which strongly motivated the analysis in this work, are related both with an alternative interpretation of the well-known *Lorenz condition*, imposed on the four-vector of electromagnetic observable potentials  $(\varphi, A) : M^4 \rightarrow T^*(M^4)$  and the classical Lagrangian formulation [97] of charged particle dynamics under external electromagnetic field. The Lagrangian approach latter is strongly dependent on an important Einsteinian notion of the rest reference frame  $\mathcal{K}_\tau$  and the related least action principle, so before explaining it in more detail, we first to analyze the classical Maxwell electromagnetic theory from a strictly dynamical point of view.

Let us consider with respect to a laboratory reference frame  $\mathcal{K}$  the additional *Lorenz condition*

$$(1.4) \quad \partial\varphi/\partial t + \langle \nabla, A \rangle = 0,$$

*a priori* assumed the Lorentz invariant wave scalar field equation

$$(1.5) \quad \partial^2\varphi/\partial t^2 - \nabla^2\varphi = \rho$$

and the charge continuity equation

$$(1.6) \quad \partial\rho/\partial t + \langle \nabla, J \rangle = 0,$$

where  $\rho : M^4 \rightarrow \mathbb{R}$  and  $J : M^4 \rightarrow \mathbb{E}^3$  are, respectively, the charge and current densities of the ambient matter. Then one can derive [125, 127] that the Lorentz invariant wave equation

$$(1.7) \quad \partial^2 A/\partial t^2 - \nabla^2 A = J$$

and the classical electromagnetic Maxwell field equations [82, 97, 57, 116, 148]

$$(1.8) \quad \begin{aligned} \nabla \times E + \partial B/\partial t &= 0, & \langle \nabla, E \rangle &= \rho, \\ \nabla \times B - \partial E/\partial t &= J, & \langle \nabla, B \rangle &= 0, \end{aligned}$$

hold for all  $(t, r) \in M^4$  with respect to the chosen laboratory reference frame  $\mathcal{K}$ .

Notice here that, inversely, Maxwell's equations (1.8) do not directly reduce, via definitions (1.2) and (1.3), to the wave field equations (1.5) and (1.7) without the Lorenz condition (1.4). This fact is very important and suggests that when it comes to a choice of governing equations, it may

be reasonable to replace Maxwell's equations (1.8) with the Lorenz condition (1.4) and the charge continuity equation (1.6). To make the equivalence statement, claimed above, more transparent we formulate it as the following proposition.

**Proposition 1.1.** *The Lorenz invariant wave equation (1.5) together with the Lorenz condition (1.4) for the observable potentials  $(\varphi, A) : M^4 \rightarrow T^*(M^4)$  and the charge continuity relationship (1.6) are completely equivalent to the Maxwell field equations (1.8).*

*Proof.* Substituting (1.4), into (1.5), one easily obtains

$$(1.9) \quad \partial^2 \varphi / \partial t^2 = - \langle \nabla, \partial A / \partial t \rangle = \langle \nabla, \nabla \varphi \rangle + \rho,$$

which implies the gradient expression

$$(1.10) \quad \langle \nabla, -\partial A / \partial t - \nabla \varphi \rangle = \rho.$$

Taking into account the electric field definition (1.2), expression (1.10) reduces to

$$(1.11) \quad \langle \nabla, E \rangle = \rho,$$

which is the second of the first pair of Maxwell's equations (1.8).

Now upon applying  $\nabla \times$  to definition (1.2), we find, owing to definition (1.3), that

$$(1.12) \quad \nabla \times E + \partial B / \partial t = 0,$$

which is the first pair of the Maxwell equations (1.8). Having differentiated with respect to the temporal variable  $t \in \mathbb{R}$  the equation (1.5) and taken into account the charge continuity equation (1.6), one finds that

$$(1.13) \quad \langle \nabla, \partial^2 A / \partial t^2 - \nabla^2 A - J \rangle = 0.$$

The latter is equivalent to the wave equation (1.7) if to observe that the current vector  $J : M^4 \rightarrow \mathbb{E}^3$  is defined by means of the charge continuity equation (1.6) up to a vector function  $\nabla \times S : M^4 \rightarrow \mathbb{E}^3$ . Now applying operation  $\nabla \times$  to the definition (1.3), owing to the wave equation (1.7) one obtains

$$(1.14) \quad \begin{aligned} \nabla \times B &= \nabla \times (\nabla \times A) = \nabla \langle \nabla, A \rangle - \nabla^2 A = \\ &= -\nabla(\partial \varphi / \partial t) - \partial^2 A / \partial t^2 + (\partial^2 A / \partial t^2 - \nabla^2 A) = \\ &= \frac{\partial}{\partial t}(-\nabla \varphi - \partial A / \partial t) + J = \partial E / \partial t + J, \end{aligned}$$

which leads directly to

$$\nabla \times B = \partial E / \partial t + J,$$

which is the first of the second pair of the Maxwell equations (1.8). The final "no magnetic charge" equation

$$\langle \nabla, B \rangle = \langle \nabla, \nabla \times A \rangle = 0,$$

in (1.8) follows directly from the elementary identity  $\langle \nabla, \nabla \times \rangle = 0$ , thereby completing the proof.  $\square$

This proposition allows us to consider the observable potential functions  $(\varphi, A) : M^4 \rightarrow T^*(M^4)$  as fundamental ingredients of the ambient *vacuum field medium*, by means of which we can try to describe the related physical behavior of charged point particles imbedded in space-time  $M^4$ . The following observation provides strong support for this approach:

**Observation.** *The Lorenz condition (1.4) actually means that the scalar potential field  $\varphi : M^4 \rightarrow \mathbb{R}$  continuity relationship, whose origin lies in some new field conservation law, characterizes the deep intrinsic structure of the vacuum field medium.*

To make this observation more transparent and precise, let us recall the definition [97, 116, 57, 148] of the electric current  $J : M^4 \rightarrow \mathbb{E}^3$  in the dynamical form

$$(1.15) \quad J := \rho u,$$

where the vector  $u \in T(\mathbb{R}^3)$  is the corresponding charge velocity. Thus, the following continuity relationship

$$(1.16) \quad \partial \rho / \partial t + \langle \nabla, \rho u \rangle = 0$$

holds, which can easily be rewritten [103] as the integral conservation law

$$(1.17) \quad \frac{d}{dt} \int_{\Omega_t} \rho(t, r) d^3 r = 0$$

for the charge inside of any bounded domain  $\Omega_t \subset \mathbb{E}^3$ , moving in the space-time  $M^4$  with respect to the natural evolution equation

$$(1.18) \quad dr/dt := u.$$

Following the above reasoning, we obtain the following result.

**Proposition 1.2.** *The Lorenz condition (1.4) is equivalent to the integral conservation law*

$$(1.19) \quad \frac{d}{dt} \int_{\Omega_t} \varphi(t, r) d^3 r = 0,$$

where  $\Omega_t \subset \mathbb{E}^3$  is any bounded domain, moving with respect to the charged point particle  $\xi$  evolution equation

$$(1.20) \quad dr/dt = u(t, r),$$

which represents the velocity vector of related local potential field changes propagating in the Minkowski space-time  $M^4$ . Moreover, for a particle with the distributed charge density  $\rho : M^4 \rightarrow \mathbb{R}$ , the following Umov type local energy conservation relationship

$$(1.21) \quad \frac{d}{dt} \int_{\Omega_t} \frac{\rho(t, r) \varphi(t, r)}{(1 - |u(t, r)|^2)^{1/2}} d^3 r = 0$$

holds for any  $t \in \mathbb{R}$ .

*Proof.* Consider first the corresponding solutions to potential field equations (1.5), taking into account condition (1.15). Owing to the standard results from [57, 97], one finds that

$$(1.22) \quad A = \varphi u,$$

which gives rise to the following form of the Lorenz condition (1.4):

$$(1.23) \quad \partial \varphi / \partial t + \langle \nabla, \varphi u \rangle = 0,$$

This obviously can be rewritten [103] as the integral conservation law (1.19), so the expression (1.19) is stated.

To state the local energy conservation relationship (1.21) it is necessary to combine the conditions (1.16), (1.23) and find that

$$(1.24) \quad \partial(\rho \varphi) / \partial t + \langle u, \nabla(\rho \varphi u) \rangle + 2\rho \varphi \langle \nabla, u \rangle = 0.$$

Taking into account that the infinitesimal volume transformation  $d^3 r = J(t, r) d^3 r_0$ , where the Jacobian  $J(t, r) := |\partial r(t; r_0) / \partial r_0|$  of the corresponding transformation  $r : \Omega_{t_0} \rightarrow \Omega_t$ , induced by the Cauchy problem for the differential relationship (1.20) for any  $t \in \mathbb{R}$ , satisfies the evolution equation

$$(1.25) \quad dJ/dt = \langle \nabla, u \rangle J,$$

easily following from (1.20), and applying to the equality (1.24) the operator  $\int_{\Omega_{t_0}} (\dots) J^2 d^3 r_0$ , one obtains that

$$(1.26) \quad \begin{aligned} \int_{\Omega_{t_0}} \frac{d}{dt} (\rho \varphi J^2) d^3 r_0 &= \frac{d}{dt} \int_{\Omega_{t_0}} (\rho \varphi J) J d^3 r_0 = \\ &= \frac{d}{dt} \int_{\Omega_t} (\rho \varphi J) d^3 r := \frac{d}{dt} \mathcal{E}(\xi; \Omega_t). \end{aligned}$$

Here we denoted the conserved charge  $\xi := \int_{\Omega_t} \rho(t, r) d^3 r$  and the local energy conservation quantity  $\mathcal{E}(\xi; \Omega_t) := \int_{\Omega_t} (\rho \varphi J) d^3 r$ . The latter quantity can be simplified, owing to the infinitesimal Lorentz invariance four-volume measure relationship  $d^3 r(t, r_0) \wedge dt = d^3 r_0 \wedge dt_0$ , where variables  $(t, r) \in \mathbb{R}_t \times \Omega_t \subset M^4$  are, within the present context, taken with respect to the moving reference frame  $\mathcal{K}_t$ , related to the infinitesimal charge quantity  $d\xi(t, r) := \rho(t, r) d^3 r$ , and variables  $(t_0, r_0) \in \mathbb{R}_{t_0} \times \Omega_{t_0} \subset M^4$  are taken with respect to the laboratory reference frame  $\mathcal{K}_{t_0}$ , related to the infinitesimal charge quantity  $d\xi(t_0, r_0) = \rho(t_0, r_0) d^3 r_0$ , satisfying the charge conservation invariance  $d\xi(t, r) =$

$d\xi(t_0, r_0)$ . The mentioned above infinitesimal Lorentz invariance relationships make it possible to calculate the local energy conservation quantity  $\mathcal{E}(\xi; \Omega_0)$  as

$$\begin{aligned}
 (1.27) \quad \mathcal{E}(\xi; \Omega_0) &= \int_{\Omega_t} (\rho\varphi J) d^3r = \int_{\Omega_t} (\rho\varphi \frac{d^3r}{d^3r_0}) d^3r = \\
 &= \int_{\Omega_t} (\rho\varphi \frac{d^3r dt}{d^3r_0 dt}) d^3r = \int_{\Omega_t} (\rho\varphi \frac{d^3r_0 dt_0}{d^3r_0 dt}) d^3r = \\
 &= \int_{\Omega_t} (\rho\varphi \frac{dt_0}{dt}) d^3r = \int_{\Omega_t} \frac{\rho\varphi d^3r}{(1-|u|^2)^{1/2}},
 \end{aligned}$$

where we took into account that  $dt = dt_0(1-|u|^2)^{1/2}$ . Thus, owing to (1.26) and (1.27) the local energy conservation relationship (1.21) is satisfied, proving the proposition.  $\square$

The constructed above local energy conservation quantity (1.27) can be rewritten as

$$(1.28) \quad \mathcal{E}(\xi; \Omega_t) = \int_{\Omega_t} \frac{d\xi(t, r)\varphi(t, r)}{(1-|u|^2)^{1/2}} := \int_{\Omega_t} d\mathcal{E}(t, r)$$

where  $d\mathcal{E}(t, r) = d\xi(t, r)\varphi(t, r)(1-|u|^2)^{-1/2}$  is the distributed in vacuum electromagnetic field energy density, related with the electric charge  $d\xi(t, r)$ , located at point  $(t, r) \in M^4$ .

The above proposition suggests a physically motivated interpretation of electrodynamic phenomena in terms of what should naturally be called *the vacuum potential field*, which determines the observable interactions between charged point particles. More precisely, we can *a priori* endow the ambient vacuum medium with a scalar potential energy field density function  $W := \xi\varphi : M^4 \rightarrow \mathbb{R}$ , where  $\xi \in \mathbb{R}_+$  is the value of an elementary charge quantity, and satisfying the governing *vacuum field equations*

$$(1.29) \quad \begin{aligned} \partial^2 W / \partial t^2 - \nabla^2 W &= \rho\xi, \quad \partial W / \partial t + \langle \nabla, \hat{A} \rangle = 0, \\ \partial^2 \hat{A} / \partial t^2 - \nabla^2 \hat{A} &= \xi\rho v, \quad \hat{A} = Wv, \end{aligned}$$

taking into account the external charged sources, which possess a virtual capability for disturbing the vacuum field medium. Moreover, this vacuum potential field function  $W : M^4 \rightarrow \mathbb{R}$  allows the natural potential energy interpretation, whose origin should be assigned not only to the charged interacting medium, but also to any other medium possessing interaction capabilities, including for instance, material particles, interacting through the gravity.

The latter leads naturally to the next important step, consisting in deriving the equation governing the corresponding potential field  $\bar{W} : M^4 \rightarrow \mathbb{R}$ , assigned to a charged point particle moving in the vacuum field medium with velocity  $u \in T(\mathbb{R}^3)$  and located at point  $r(t) = R(t) \in \mathbb{E}^3$  at time  $t \in \mathbb{R}$ . As can be readily shown [125, 126, 136], the corresponding evolution equation governing the related potential field function  $\bar{W} : M^4 \rightarrow \mathbb{R}$ , assigned to a moving in the space  $\mathbb{E}^3$  charged particle  $\xi$  has the form

$$(1.30) \quad \frac{d}{dt}(-\bar{W}u) = -\nabla\bar{W},$$

where  $\bar{W} := W(r, t)|_{r \rightarrow R(t)}$ ,  $u(t) := dR(t)/dt$  at point particle location  $(t, R(t)) \in M^4$ .

Similarly, if there are two interacting charged point particles, located at points  $r(t) = R(t)$  and  $r_f(t) = R_f(t) \in \mathbb{E}^3$  at time  $t \in \mathbb{R}$  and moving, respectively, with velocities  $u := dR(t)/dt$  and  $u_f := dR_f(t)/dt$ , the corresponding potential field function  $\bar{W}' : M^4 \rightarrow \mathbb{R}$ , considered with respect to the reference frame  $\mathcal{K}'$  specified by Euclidean coordinates  $(t', r - r_f) \in \mathbb{E}^4$  and moving with the velocity  $u_f \in T(\mathbb{R}^3)$  subject to the laboratory reference frame  $\mathcal{K}$ , should satisfy [125, 126] the dynamical equality

$$(1.31) \quad \frac{d}{dt'}[-\bar{W}'(u' - u'_f)] = -\nabla\bar{W}',$$

where, by definition, we have denoted the velocity vectors  $u' := dr/dt'$ ,  $u'_f := dr_f/dt' \in T(\mathbb{R}^3)$ . The dynamical potential field equations (1.30) and (1.31) appear to have important properties and can be used as means for representing classical electrodynamic phenomena. Consequently, we shall proceed to investigate their physical properties in more detail and compare them with classical results for Lorentz type forces arising in the electrodynamics of moving charged point particles in an external electromagnetic field.

In this investigation, we were in part inspired by works [38, 48, 155] and interesting studies [37, 11, 8, 36] devoted to solving the classical problem of reconciling gravitational and electrodynamic charges within the Mach-Einstein ether paradigm. First, we will revisit the classical Mach-Einstein relativistic electrodynamics of a moving charged point particle, and second, we study the resulting electrodynamic theories associated with our vacuum potential field dynamical equations (1.30) and (1.31), making use of the fundamental Lagrangian and Hamiltonian formalisms which were specially devised in [19, 127].

**1.1.1. Classical relativistic electrodynamics revisited.** The classical relativistic electrodynamics of a freely moving charged point particle in the Minkowski space-time  $M^4 \simeq \mathbb{R} \times \mathbb{E}^3$  is based on the Lagrangian approach [97, 57, 116, 148, 39] with Lagrangian function

$$(1.32) \quad \mathcal{L}_0 := -m_0(1 - |u|^2)^{1/2},$$

where  $m_0 \in \mathbb{R}_+$  is the so-called particle rest mass parameter and  $u \in T(\mathbb{R}^3)$  is its spatial velocity in the Euclidean space  $\mathbb{E}^3$ , expressed here and in the sequel in light speed units (with light speed  $c = 1$ ). The least action principle in the form

$$(1.33) \quad \delta S = 0, \quad S := - \int_{t_1}^{t_2} m_0(1 - |u|^2)^{1/2} dt$$

for any fixed temporal interval  $[t_1, t_2] \subset \mathbb{R}$  gives rise to the well-known relativistic relationships for the mass of the particle

$$(1.34) \quad m = m_0(1 - |u|^2)^{-1/2},$$

the momentum of the particle

$$(1.35) \quad p := mu = m_0 u(1 - |u|^2)^{-1/2}$$

and the energy of the particle

$$(1.36) \quad \mathcal{E}_0 = m = m_0(1 - |u|^2)^{-1/2}.$$

It follows from [97, 116], that the origin of the Lagrangian (1.32) can be extracted from the action

$$(1.37) \quad S := - \int_{t_1}^{t_2} m_0(1 - |u|^2)^{1/2} dt = - \int_{\tau_1}^{\tau_2} m_0 d\tau,$$

on the suitable temporal interval  $[\tau_1, \tau_2] \subset \mathbb{R}$ . Here  $m_0 \in \mathbb{R}_+$  is considered as a constant positive parameter *a priori* attributed to the point particle,

$$(1.38) \quad d\tau := dt(1 - |u|^2)^{1/2}$$

and  $\tau \in \mathbb{R}$  is the so-called, proper temporal parameter assigned to a freely moving particle with respect to the rest reference frame  $\mathcal{K}_\tau$ . The action (1.37) is rather questionable from the dynamical point of view, since it is physically defined with respect to the rest reference frame  $\mathcal{K}_\tau$ , giving rise to the constant action  $S = -m_0(\tau_2 - \tau_1)$ , as the limits of integrations  $\tau_1 < \tau_2 \in \mathbb{R}$  were taken to be fixed from the very beginning. Moreover, considering this particle to have a charge  $\xi \in \mathbb{R}$  and be moving in the Minkowski space-time  $M^4$  under action of an external electromagnetic field  $(\varphi, A) \in M^4$ , the corresponding classical (relativistic) action functional is chosen (see [97, 57, 10, 9, 116, 148, 19, 127]) with respect to the *rest reference system*  $\mathcal{K}_\tau$  as follows:

$$(1.39) \quad S := \int_{\tau_1}^{\tau_2} [-m_0 d\tau + \xi \langle A, \dot{r} \rangle d\tau - \xi \varphi(1 + |\dot{r}|^2)^{-1/2} d\tau],$$

being parameterized by the *Euclidean* space-time variables  $(\tau, r) \in \mathbb{E}^4$  satisfying the infinitesimal relationship  $d\tau^2 + |dr|^2 = dt^2$ , where we have denoted  $\dot{r} := dr/d\tau$  in contrast to the definition  $u := dr/dt$ . The action (1.39) can be rewritten with respect to the laboratory reference frame  $\mathcal{K}$  as

$$(1.40) \quad S = \int_{t_1}^{t_2} \mathcal{L} dt, \quad \mathcal{L} := -m_0(1 - |u|^2)^{1/2} + \xi \langle A, u \rangle - \xi \varphi,$$

defined on the suitable temporal interval  $[t_1, t_2] \subset \mathbb{R}$ . The action function (1.40) contains two physically incompatible sub-integral parts - the first one  $-m_0(1 - |u|^2)^{1/2}dt = -m_0d\tau$ , having sense with respect to the rest reference frame  $\mathcal{K}_\tau$  and the second one, equivalent to  $\langle A, dr \rangle - \xi\varphi dt$ , having sense with respect to the laboratory reference frame  $\mathcal{K}$ . Nonetheless, the least action principle applied to the functional (1.40) gives rise to the following [97, 57, 116, 148] dynamical equation

$$(1.41) \quad dP/dt = -\nabla(\xi\varphi - \langle \xi A, u \rangle),$$

where, by definition, the generalized particle-field momentum

$$(1.42) \quad P = p + \xi A,$$

the own particle momentum

$$(1.43) \quad p = mu = m_0u(1 - |u|^2)^{-1/2}$$

and its so called "inertial" mass

$$(1.44) \quad m = m_0(1 - |u|^2)^{-1/2}.$$

The corresponding particle conserved energy equals

$$(1.45) \quad \mathcal{E} = (m_0^2 + |p|^2)^{1/2} + \xi\varphi,$$

that is

$$(1.46) \quad d\mathcal{E}/dt = 0 = d\mathcal{E}/d\tau$$

with respect to both the laboratory reference frame  $\mathcal{K}$  and the rest reference frame  $\mathcal{K}_\tau$ .

The obtained above expression (1.45) for the particle energy  $\mathcal{E} \in \mathbb{R}$  appears to be open to question, since the electrical potential energy  $\xi\varphi$ , entering additively, has no affect on the relativistic particle mass  $m = m_0(1 - |u|^2)^{-1/2}$ , contradicting the experimental facts [57, 82] that some part of the observable charged particle mass is of the electromagnetic origin. This fact was also underlined by L. Brillouin [33], who remarked that the fact that the potential energy has no affect on the particle mass tells us that "... *any possibility of existence of a particle mass related with an external potential energy, is completely excluded*". Moreover, it is necessary to stress here that the least action principle, based on the action functional (1.40) and formulated with respect to the laboratory reference frame  $\mathcal{K}$  time parameter  $t \in \mathbb{R}$ , appears to be logically inadequate, for there is a strong physical inconsistency with other time parameters of the Lorentz equivalent laboratory reference frames depending simultaneously both on the spatial and temporal coordinates. This was first mentioned by R. Feynman in [57], in his efforts to rewrite the Lorentz force expression with respect to the rest reference frame  $\mathcal{K}_\tau$ . This and other special relativity theory and electrodynamics problems stimulated many prominent physicists of the past [33, 57, 154, 116, 32] and present [16, 108, 107, 155, 38, 48, 100, 101, 136, 114, 70, 36] to try to develop alternative relativity theories based on completely different space-time and matter structure principles. In particular, in [16] authors have analyzed a super-symmetric version of the classical relativistic charged particle electrodynamics and shown that its physically reasonable least action formulation is achieved only by means of the reformulation of the corresponding Lagrangian function with respect to the proper rest reference frame  $\mathcal{K}_\tau$ .

There also is another controversial inference from the action expression (1.40) and resulting dynamical equation (1.41): the force  $F_\xi = dP/dt$ , exerted by the external electromagnetic field on the *particle-field cluster* [57] carrying the momentum  $P = p + \xi A$ , appears to be the standard gradient expression

$$(1.47) \quad F_\xi = -\nabla W_\xi,$$

where the generalized "potential energy"

$$(1.48) \quad W_\xi := \xi\varphi - \langle \xi A, u \rangle.$$

Its first part  $\xi\varphi \in \mathbb{R}$  equals the classical [57, 82] electrical potential energy, but its second part  $-\langle \xi A, u \rangle$  is strictly related with magnetic vector potential  $A \in \mathbb{E}^3$  and has nowadays no reasonable physical explanation. As one can easily show [97, 116, 57, 148, 20] from (1.41), the corresponding expression for the *classical Lorentz force* is given as

$$(1.49) \quad dp/dt = F := \xi E + \xi u \times B,$$

where we have defined, as before,

$$(1.50) \quad E := -\partial A / \partial t - \nabla \varphi$$

for the corresponding electric field and

$$(1.51) \quad B := \nabla \times A$$

for the related magnetic field, acting on the point particle with the electric charge  $\xi \in \mathbb{R}$ . The expression (1.49) means, in particular, that the Lorentz force (1.49) depends linearly on the particle velocity vector  $u \in T(\mathbb{R}^3)$ , and so there is a strong dependence on the reference frame with respect to which the charged point particle  $\xi$  moves. Attempts to reconcile this and some related controversies [33, 57, 136, 88] forced Einstein to devise his special relativity theory and proceed further to creating his general relativity theory trying to explain the gravity by means of geometrization of space-time and matter in the Universe.

Here we once more mention that the classical Lagrangian function  $\mathcal{L} : T(M^4) \rightarrow \mathbb{R}$  in (1.40) is simultaneously written as a combination of terms incompatibly expressed from the physical point of view by means of both the Euclidean rest reference frame variables  $(\tau, r) \in \mathbb{E}^4$ , naturally attributed to the charged point particle, and arbitrarily chosen Minkowski reference frame variables  $(t, r) \in M^4$ .

These problems were recently analyzed within a completely different "no-geometry" approach [125, 126], where new dynamical equations were derived, which were free of the controversial elements mentioned above. Moreover, this approach avoided the introduction of the well known Lorentz transformations of the space-time reference frames with respect to which the action functional (1.40) is invariant. From this point of view, there are interesting for discussion of conclusions in [140, 72, 5, 8, 11], and where some electrodynamic models, possessing intrinsic Galilean and Poincaré-Lorentz symmetries, are reanalyzed from diverse geometrical points of view. Subject to a possible geometric space-type structure and the related vacuum field background, exerting the decisive influence on the particle dynamics, we need to mention here recent works [6, 145] and the closely related with their ideas the classical articles [86, 118]. Next, we shall revisit the results obtained recently in [125, 126, 125, 20] from the classical Lagrangian and Hamiltonian formalisms [19] in order to shed new light on the physical underpinnings of the vacuum field theory approach to the study of combined electromagnetic and, eventually, also gravitational effects.

## 1.2. The vacuum field theory electrodynamics equations: Lagrangian analysis.

1.2.1. *A point particle moving in vacuum - an alternative electrodynamic model.* Within the vacuum field theory approach to electromagnetism, devised in [125, 126], the main vacuum potential field function  $\bar{W} : M^4 \rightarrow \mathbb{R}$ , related to a charged point particle  $\xi$ , satisfies the differential evolution equation (1.30), namely

$$(1.52) \quad \frac{d}{dt}(-\bar{W}u) = -\nabla \bar{W},$$

in the case when all of the external charged particles are at rest, that is  $\partial \bar{W} / \partial t = 0$ , and as above,  $u := dr/dt$  is the particle velocity with respect to some laboratory reference system  $\mathcal{K}$ , specified by the Minkowski coordinates  $(t, r) \in M^4$ .

To analyze the dynamical equation (1.52) from the Lagrangian point of view, we write the corresponding action functional as

$$(1.53) \quad S := - \int_{t_1}^{t_2} \bar{W} dt = - \int_{\tau_1}^{\tau_2} \bar{W} (1 + |\dot{r}|^2)^{1/2} d\tau,$$

expressed with respect to the rest reference frame  $\mathcal{K}_\tau$ , specified by the Euclidean coordinates  $(\tau, r) \in \mathbb{E}^4$ . Fixing the proper temporal parameters  $\tau_1 < \tau_2 \in \mathbb{R}$ , one finds from the least action principle  $\delta S = 0$  that

$$(1.54) \quad \begin{aligned} p &:= \partial \mathcal{L} / \partial \dot{r} = -\bar{W} \dot{r} (1 + |\dot{r}|^2)^{-1/2} = -\bar{W} u, \\ \dot{p} &:= dp/d\tau = \partial \mathcal{L} / \partial r = -\nabla \bar{W} (1 + |\dot{r}|^2)^{1/2}, \end{aligned}$$

where, owing to (1.53), the corresponding Lagrangian function is

$$(1.55) \quad \mathcal{L} := -\bar{W} (1 + |\dot{r}|^2)^{1/2}.$$



Recalling now the definition of the particle "inertial" mass

$$(1.56) \quad m := -\bar{W}$$

and the relationships

$$(1.57) \quad d\tau = dt(1 - |u|^2)^{1/2} = dt(1 + |\dot{r}|^2)^{-1/2}, \quad \dot{r}d\tau = udt,$$

from (1.54) we easily obtain the classical dynamical equation exactly coinciding with (1.52):

$$(1.58) \quad dp/dt = -\nabla\bar{W}.$$

Moreover, one now readily finds that the corresponding dynamical mass, defined by means of expression (1.56), is given as

$$(1.59) \quad m = m_0(1 - |u|^2)^{-1/2}, \quad m_0 := -\bar{W}(R(t_0)),$$

where  $u(t)|_{t=t_0} = 0$  at the spatial point  $r = R(t_0) \in \mathbb{E}^3$ , and which completely coincides with expression (1.34) of the preceding section. Now one can formulate the following proposition using the results obtained above.

**Proposition 1.3.** *The alternative freely moving point particle electrodynamic model (1.52) allows the physically reasonable least action formulation based on the action functional (1.53) with respect to the "rest" reference frame variables, where the Lagrangian function is given by expression (1.55). The related electrodynamics is completely equivalent to that of a classical relativistic freely moving point particle, described in Subsection 1.1.*

1.2.2. *A moving in vacuum interacting two charge system - an alternative electrodynamic model.*

We proceed now to the case when our charged point particle  $\xi$  moves in the space-time with velocity vector  $u \in T(\mathbb{R}^3)$  and interacts with another external charged point particle  $\xi_f$ , moving with velocity vector  $u_f \in T(\mathbb{R}^3)$  with respect to a common reference frame  $\mathcal{K}$ . As was shown in [125, 126], the respectively modified dynamical equation for the vacuum potential field function  $\bar{W}' : M^4 \rightarrow \mathbb{R}$  subject to the moving reference frame  $\mathcal{K}'$  is given by equality (1.31), or

$$(1.60) \quad \frac{d}{dt'}[-\bar{W}'(u' - u'_f)] = -\nabla\bar{W}',$$

where, as before, the velocity vectors  $u' := dr/dt'$ ,  $u'_f := dr_f/dt' \in T(\mathbb{R}^3)$ . Since the external charged particle  $\xi_f$  moves in the space-time  $M^4$ , it generates the related magnetic field  $B := \nabla \times A$ , whose magnetic vector potential  $A : M^4 \rightarrow \mathbb{E}^3$  is defined, owing to the results of [125, 126, 136], as

$$(1.61) \quad \xi A := \bar{W}u_f.$$

Whence, taking into account that the field potential

$$(1.62) \quad \bar{W} = \bar{W}'(1 - |u_f|^2)^{-1/2}$$

and the particle momentum  $p' = -\bar{W}'u' = -\bar{W}u$ , equality (1.60) becomes equivalent to

$$(1.63) \quad \frac{d}{dt'}(p' + \xi A') = -\nabla\bar{W}',$$

if considered with respect to the moving reference frame  $\mathcal{K}'$ , or to the Lorentz type force equality

$$(1.64) \quad \frac{d}{dt}(p + \xi A) = -\nabla\bar{W}(1 - |u_f|^2),$$

if considered with respect to the laboratory reference frame  $\mathcal{K}$ , owing to the classical Lorentz invariance relationship (1.62), as the corresponding magnetic vector potential, generated by the external charged point test particle  $\xi_f$  with respect to the reference frame  $\mathcal{K}'$ , is identically equal to zero. To imbed the dynamical equation (1.64) into the classical Lagrangian formalism, we start from the following action functional, which naturally generalizes the functional (1.53):

$$(1.65) \quad S := - \int_{\tau_1}^{\tau_2} \bar{W}'(1 + |\dot{r} - \dot{r}_f|^2)^{1/2} d\tau.$$

Here, as before,  $\bar{W}'$  is the respectively calculated vacuum field potential  $\bar{W}$  subject to the moving reference frame  $\mathcal{K}'$ ,  $\dot{r} = u'dt'/d\tau$ ,  $\dot{r}_f = u'_fdt'/d\tau$ ,  $d\tau = dt'(1 - |u' - u'_f|^2)^{1/2}$ , which take into account the relative velocity of the charged point particle  $\xi$  subject to the reference frame  $\mathcal{K}'$ , specified by the Euclidean coordinates  $(t', r - r_f) \in \mathbb{R}^4$ , and moving simultaneously with velocity

vector  $u_f \in T(\mathbb{R}^3)$  with respect to the laboratory reference frame  $\mathcal{K}$ , specified by the Minkowski coordinates  $(t, r) \in M^4$  and related to those of the reference frame  $\mathcal{K}'$  and  $\mathcal{K}_\tau$  by means of the following infinitesimal relationships:

$$(1.66) \quad dt^2 = (dt')^2 + |dr_f|^2, \quad (dt')^2 = d\tau^2 + |dr - dr_f|^2.$$

So, it is clear in this case that our charged point particle  $\xi$  moves with the velocity vector  $u' - u'_f \in T(\mathbb{R}^3)$  with respect to the reference frame  $\mathcal{K}'$  in which the external charged particle  $\xi_f$  is at rest. Thereby, we have reduced the problem of deriving the charged point particle  $\xi$  dynamical equation to that before solved in Subsection 1.2.1.

Now we can compute the least action variational condition  $\delta S = 0$ , taking into account that, owing to (1.65), the corresponding Lagrangian function with respect to the rest reference frame  $\mathcal{K}_\tau$  is given as

$$(1.67) \quad \mathcal{L} := -\bar{W}'(1 + |\dot{r} - \dot{r}_f|^2)^{1/2}.$$

As a result of simple calculations, the generalized momentum of the charged particle  $\xi$  equals

$$(1.68) \quad \begin{aligned} P := \partial \mathcal{L} / \partial \dot{r} &= -\bar{W}'(\dot{r} - \dot{r}_f)(1 + |\dot{r} - \dot{r}_f|^2)^{-1/2} = \\ &= -\bar{W}'\dot{r}(1 + |\dot{r} - \dot{r}_f|^2)^{-1/2} + \bar{W}'\dot{r}_f(1 + |\dot{r} - \dot{r}_f|^2)^{-1/2} = \\ &= mu' + \xi A' := p' + \xi A' = p + \xi A, \end{aligned}$$

where, owing to (1.62) the vectors  $p' := -\bar{W}'u' = -\bar{W}u = p \in \mathbb{E}^3$ ,  $A' = \bar{W}'u'_f = \bar{W}u_f = A \in \mathbb{E}^3$ , and giving rise to the dynamical equality

$$(1.69) \quad \frac{d}{d\tau}(p' + \xi A') = -\nabla \bar{W}'(1 + |\dot{r} - \dot{r}_f|^2)^{1/2}$$

with respect to the rest reference frame  $\mathcal{K}_\tau$ . As  $dt' = d\tau(1 + |\dot{r} - \dot{r}_f|^2)^{1/2}$  and  $(1 + |\dot{r} - \dot{r}_f|^2)^{1/2} = (1 - |u' - u'_f|^2)^{-1/2}$ , we obtain from (1.69) the equality

$$(1.70) \quad \frac{d}{dt'}(p' + \xi A') = -\nabla \bar{W}',$$

exactly coinciding with equality (1.63) subject to the moving reference frame  $\mathcal{K}'$ . Now, making use of expressions (1.66) and (1.62), one can rewrite (1.70) as that with respect to the laboratory reference frame  $\mathcal{K}$ :

$$(1.71) \quad \begin{aligned} \frac{d}{dt'}(p' + \xi A') &= -\nabla \bar{W}' \Rightarrow \\ &\Rightarrow \frac{d}{dt'}\left(\frac{-\bar{W}u'}{(1+|u'_f|^2)^{1/2}} + \frac{\xi \bar{W}u'_f}{(1+|u'_f|^2)^{1/2}}\right) = -\frac{\nabla \bar{W}}{(1+|u'_f|^2)^{1/2}} \Rightarrow \\ &\Rightarrow \frac{d}{dt'}\left(\frac{-\bar{W}dr}{(1+|u'_f|^2)^{1/2}dt'} + \frac{\xi \bar{W}dr_f}{(1+|u'_f|^2)^{1/2}}\right) = -\frac{\nabla \bar{W}}{(1+|u'_f|^2)^{1/2}} \Rightarrow \\ &\Rightarrow \frac{d}{dt}\left(-\bar{W}\frac{dr}{dt} + \xi \bar{W}\frac{dr_f}{dt}\right) = -\nabla \bar{W}(1 - |u_f|^2), \end{aligned}$$

exactly coinciding with (1.64):

$$(1.72) \quad \frac{d}{dt}(p + \xi A) = -\nabla \bar{W}(1 - |u_f|^2).$$

*Remark 1.4.* The equation (1.72) allows to infer the following important and physically reasonable phenomenon: if the test charged point particle velocity  $u_f \in T(\mathbb{R}^3)$  tends to the light velocity  $c = 1$ , the corresponding acceleration force  $F_{ac} := -\nabla \bar{W}(1 - |u_f|^2)$  is vanishing. Thereby, the electromagnetic fields, generated by such rapidly moving charged point particles, have no influence on the dynamics of charged objects if observed with respect to an arbitrarily chosen laboratory reference frame  $\mathcal{K}$ .

The latter equation (1.72) can be easily rewritten as

$$(1.73) \quad \begin{aligned} dp/dt &= -\nabla \bar{W} - \xi dA/dt + \nabla \bar{W}|u_f|^2 = \\ &= \xi(-\xi^{-1}\nabla \bar{W} - \partial A/\partial t) - \xi \langle u, \nabla \rangle A + \xi \nabla \langle A, u_f \rangle, \end{aligned}$$

or, using the well-known [97] identity

$$(1.74) \quad \nabla \langle a, b \rangle = \langle a, \nabla \rangle b + \langle b, \nabla \rangle a + b \times (\nabla \times a) + a \times (\nabla \times b),$$

where  $a, b \in \mathbb{E}^3$  are arbitrary vector functions, in the standard Lorentz type form

$$(1.75) \quad dp/dt = \xi E + \xi u \times B - \nabla \langle \xi A, u - u_f \rangle.$$

The result (1.75), being before found with respect to the moving reference frame  $\mathcal{K}'$  in [125, 126, 136] and [105], makes it possible to formulate the next important proposition.

**Proposition 1.5.** *The alternative classical relativistic electrodynamic model (1.63) allows the least action formulation based on the action functional (1.65) with respect to the rest reference frame  $\mathcal{K}_\tau$ , where the Lagrangian function is given by expression (1.67). The resulting Lorentz type force expression equals (1.75), being modified by the additional force component  $F_c := -\nabla \langle \xi A, u - u_f \rangle$ , important for explanation [2, 31, 149] of the well known Aharonov-Bohm effect.*

1.2.3. *A moving charged point particle formulation dual to the classical alternative electrodynamic model.* It is easy to see that the action functional (1.65) is written utilizing the standard classical Lorentz transformations of reference frames. If we now consider the action functional (1.53) for a charged point particle moving with respect to the rest reference frame  $\mathcal{K}_\tau$ , and take into account its interaction with an external magnetic field generated by the vector potential  $A : M^4 \rightarrow \mathbb{E}^3$ , it can be naturally generalized as

$$(1.76) \quad S := \int_{t_1}^{t_2} (-\bar{W} dt + \xi \langle A, dr \rangle) = \int_{\tau_1}^{\tau_2} (-\bar{W}(1 + |\dot{r}|^2)^{1/2} + \xi \langle A, \dot{r} \rangle) d\tau,$$

where  $d\tau = dt(1 - |u|^2)^{1/2}$ . The chosen form of functional (1.76) can be explained by means of the following physically motivated reasonings. Consider an action functional like (1.76) and calculate its value along any smooth arbitrarily chosen and dynamically admissible closed path  $l \subset M^4$ , which should be naturally put to be zero:

$$(1.77) \quad 0 = \int_l (-\bar{W} dt + \xi \langle A, dr \rangle).$$

Having applied to the right-hand side of (1.77) the standard Stokes theorem [1], one easily obtains that

$$(1.78) \quad \begin{aligned} \oint_l (-\bar{W} dt + \xi \langle A, dr \rangle) &= \int_{S(l)} (-\langle \nabla \bar{W}, dr \wedge dt \rangle - \\ &- \langle \xi \partial A / \partial t, dr \wedge dt \rangle) = \int_{S(l)} \langle -\nabla \bar{W} - \xi \partial A / \partial t, dr \wedge dt \rangle = \\ &= \int_{S(l)} \langle \xi E, dr \wedge dt \rangle = \int_{S(l)} \langle F, dr \wedge dt \rangle = \\ &= \int_{S(l)} \langle F_\xi, dr \wedge dt \rangle = - \int_{S(l)} \langle d\mathcal{E} \wedge dt \rangle = \\ &= - \int_{S(l)} d(\mathcal{E} dt) = - \oint_l \mathcal{E} dt = 0, \end{aligned}$$

if and only if the charged point particle energy  $\mathcal{E}$  is conserved along this arbitrarily chosen and admissible path  $l \subset M^4$ . As a simple consequence of (1.77) the work performed by the electromagnetic force  $F_\xi$  depends only on the electric field  $E \in \mathbb{E}^3$ , not depending on the related magnetic field  $B = \nabla \times A \in \mathbb{E}^3$ . Thus, having assumed that the corresponding charged point particle dynamical equations conform the energy conservation condition mentioned above, the action functional (1.76) can be accepted as reasonable from physical point of view.

*Remark 1.6.* It is also interesting to remark that a condition  $\int_l \mathcal{L} dt = 0$ , similar to (1.77), calculated for the Lagrangian function  $\mathcal{L} = \frac{m|\dot{r}|^2}{2} - \bar{W}$  in the classical mechanics of a point particle with mass

$m \in \mathbb{R}_+$  moving under an external potential  $\bar{W} : \mathbb{R}^3 \rightarrow \mathbb{R}$ , gives rise to the true classical Newton's mechanics:

$$\begin{aligned}
 (1.79) \quad & \oint_l \left( \frac{m|\dot{r}|^2}{2} - \bar{W} \right) dt = \oint_l \left( \frac{m}{2} \langle \dot{r}, \dot{r} \rangle - \bar{W} \right) dt = \\
 & = \int_{S(l)} \left( \langle m\dot{r}, d\dot{r} \wedge dt \rangle - \langle \nabla \bar{W}, dr \wedge dt \rangle \right) = \\
 & = \int_{S(l)} \left( - \langle mdr, \wedge d\dot{r} \rangle - \langle \nabla \bar{W}, dr \wedge dt \rangle \right) = \\
 & = \int_{S(l)} \left( - \langle dr, \wedge m\ddot{r}dt \rangle - \langle \nabla \bar{W}, dr \wedge dt \rangle \right) = \\
 & = - \int_{S(l)} \left( \langle m\ddot{r}, dr \wedge dt \rangle + \langle \nabla \bar{W}, dr \wedge dt \rangle \right) = \\
 & = - \int_{S(l)} \langle m\ddot{r} + \nabla \bar{W}, dr \wedge dt \rangle = 0
 \end{aligned}$$

if and only if the Newton's equation

$$(1.80) \quad m\ddot{r} = -\nabla \bar{W}$$

holds.

The least action condition  $\delta S = 0$ , as calculated with respect to the rest reference frame  $\mathcal{K}_\tau$ , states in the Feynman's spirit [57] of reasonings that the charged point particle  $\xi$  chooses in the Minkowski space-time  $M^4$  such a trajectory of its motion, which realizes the least action value of the functional (1.76), calculated namely with respect to its own rest reference time parameter  $\tau \in \mathbb{R}$ , being *a unique physically sensible quantity attributed to the charged point particle dynamics*. Really, as it was stressed by R. Feynman [57], the least action principle, as applied to the functional (1.76) with respect to the laboratory reference frame time parameter, gives rise to a senseless expression, whose value is both ambiguous and physically not well defined. Thus, the corresponding common generalized particle-field momentum takes the form

$$\begin{aligned}
 (1.81) \quad & P := \partial \mathcal{L} / \partial \dot{r} = -\bar{W} \dot{r} (1 + |\dot{r}|^2)^{-1/2} + \xi A = \\
 & = mu + \xi A := p + \xi A,
 \end{aligned}$$

and satisfies the equation

$$\begin{aligned}
 (1.82) \quad & \dot{P} := dP/d\tau = \partial \mathcal{L} / \partial r = -\nabla \bar{W} (1 + |\dot{r}|^2)^{1/2} + \xi \nabla \langle A, \dot{r} \rangle = \\
 & = -\nabla \bar{W} (1 - |u|^2)^{-1/2} + \xi \nabla \langle A, u \rangle (1 - |u|^2)^{-1/2},
 \end{aligned}$$

where

$$(1.83) \quad \mathcal{L} := -\bar{W} (1 + |\dot{r}|^2)^{1/2} + \xi \langle A, \dot{r} \rangle$$

is the corresponding Lagrangian function. Since  $d\tau = dt(1 - |u|^2)^{1/2}$ , one easily finds from (1.82) that

$$(1.84) \quad dP/dt = -\nabla(\bar{W} - \xi \langle A, u \rangle).$$

Upon substituting (1.81) into (1.84) and making use of the identity (1.74), we obtain the classical expression for the Lorentz force  $F_\xi$ , acting on the moving charged point particle  $\xi$ :

$$(1.85) \quad dp/dt := F_\xi = \xi E + \xi u \times B,$$

where, by definition,

$$(1.86) \quad E := -\xi^{-1} \nabla \bar{W} - \partial A / \partial t$$

is its associated electric field and

$$(1.87) \quad B := \nabla \times A$$

is the corresponding magnetic field. This wondering result can be summarized as follows.

**Proposition 1.7.** *The classical relativistic Lorentz force (1.85) allows the least action formulation based on the action functional (1.76) with respect to the rest reference frame  $\mathcal{K}_\tau$ , where the Lagrangian function is given by formula (1.83).*

Concerning the related electrodynamics of a charged point particle  $\xi$ , described by the dual classical Lorentz force (1.85), we need to state that it is *not equivalent* to that described by means of the classical Lorentz force (1.49). Moreover, one can easily observe that the classical Lorentz force  $F_\xi = \xi E + \xi u \times B$ , exerted on the charged point particle  $\xi$  by an external charged point test particle  $\xi_f$  is not *a priori* vanishing as it should follow from the relativistic physics point of view. The details of these aspects will be analyzed in more details in the next Section to follow.

Comparing the obtained above Lorentz type forces expressions (1.85) and (1.75), differing by the gradient term  $F_c := -\xi \nabla < A, u - u_f >$ , which reconciles the dual Lorentz force acting on a moving charged point particle  $\xi$  with respect to an arbitrarily chosen laboratory reference frames  $\mathcal{K}$  and, as it is was mentioned below, is responsible [2, 31, 149] for the Aharonov-Bohm effect. This fact is important for our vacuum field theory approach since it uses no special geometry and makes it possible to analyze electromagnetic and, under some conditions, also gravitational fields simultaneously by employing the new definition of the dynamical mass by means of expression (1.56).

**1.3. The vacuum field theory electrodynamics equations: Hamiltonian analysis.** Any Lagrangian theory has an equivalent canonical Hamiltonian representation via the classical Legendre transformation [7, 148, 1, 119, 20]. As we have already formulated our vacuum field theory of a moving particle with a charge  $\xi \in \mathbb{R}$  in Lagrangian form, we proceed now to its Hamiltonian analysis making use of the action functionals (1.53), (1.67) and (1.76).

Take, first, the Lagrangian function (1.55) and the momentum expression (1.54) for defining the corresponding Hamiltonian function

$$\begin{aligned} H &:= < p, \dot{r} > - \mathcal{L} = \\ &= -|p|^2 \bar{W}^{-1} (1 - |p|^2 / \bar{W}^2)^{-1/2} + \bar{W} (1 - |p|^2 / \bar{W}^2)^{-1/2} = \\ (1.88) \quad &= -|p|^2 \bar{W}^{-1} (1 - |p|^2 / \bar{W}^2)^{-1/2} + \bar{W}^2 \bar{W}^{-1} (1 - |p|^2 / \bar{W}^2)^{-1/2} = \\ &= -(\bar{W}^2 - |p|^2)(\bar{W}^2 - |p|^2)^{-1/2} = -(\bar{W}^2 - |p|^2)^{1/2}. \end{aligned}$$

Consequently, it is easy to show [1, 7, 148, 119] that the Hamiltonian function (1.88) is a conservation law of the dynamical field equation (1.52); that is, for all  $\tau, t \in \mathbb{R}$

$$(1.89) \quad dH/dt = 0 = dH/d\tau,$$

which naturally leads to an energy interpretation of  $H$ . Thus, we can represent the particle energy as

$$(1.90) \quad \mathcal{E} = (\bar{W}^2 - |p|^2)^{1/2}.$$

The corresponding Hamiltonian system equivalent to the vacuum field equation (1.52) can be written as

$$\begin{aligned} (1.91) \quad \dot{r} &:= dr/d\tau = \partial H / \partial p = p(\bar{W}^2 - |p|^2)^{-1/2} \\ \dot{p} &:= dp/d\tau = -\partial H / \partial r = \bar{W} \nabla \bar{W} (\bar{W}^2 - |p|^2)^{-1/2}, \end{aligned}$$

and we have the following result.

**Proposition 1.8.** *The alternative freely moving point particle electrodynamic model, based on the action functional (1.53), allows the canonical Hamiltonian formulation (1.91) with respect to the rest reference frame  $\mathcal{K}$ , where the Hamiltonian function is given by expression (1.88).*

Concerning the charged point particle electrodynamics, based on the dynamical equations (1.91), we state that it is completely equivalent to the classical relativistic freely moving point particle electrodynamics described above in Subsection 1.1.

In an analogous manner, one can now use the Lagrangian (1.67) and equation (1.82) to construct the Hamiltonian function for the dynamical field equation (1.64), describing the motion of a charged point particle  $\xi$  in an external electromagnetic field as

$$(1.92) \quad \dot{r} := dr/d\tau = \partial H / \partial P, \quad \dot{P} := dP/d\tau = -\partial H / \partial r,$$

where, by definition,

$$\begin{aligned}
 H &:= \langle P, \dot{r} \rangle - \mathcal{L} = \\
 &= \langle P, \dot{r}_f - P\bar{W}'^{-1}(1 - |P|^2/\bar{W}'^2)^{-1/2} \rangle + \bar{W}'[\bar{W}'^2(\bar{W}'^2 - |P|^2)^{-1}]^{1/2} = \\
 &= \langle P, \dot{r}_f \rangle + |P|^2(\bar{W}'^2 - |P|^2)^{-1/2} - \bar{W}'^2(\bar{W}'^2 - |P|^2)^{-1/2} = \\
 (1.93) \quad &= -(\bar{W}'^2 - |P|^2)(\bar{W}'^2 - |P|^2)^{-1/2} + \langle P, \dot{r}_f \rangle = \\
 &= -(\bar{W}'^2 - |P|^2)^{1/2} - \langle \xi A, P \rangle (\bar{W}'^2 - |P|^2)^{-1/2}.
 \end{aligned}$$

Here we took into account that, owing to definitions (1.61) and (1.68),

$$\begin{aligned}
 (1.94) \quad \xi_f A' &:= \bar{W}' u'_f = \bar{W}' dr_f/dt' = \bar{W} u_f = \xi_f A = \\
 &= \bar{W}' \frac{dr_f}{d\tau} \cdot \frac{d\tau}{dt'} = \bar{W} \dot{r}_f (1 - |u' - u'_f|)^{1/2} = \\
 &= \bar{W}' \dot{r}_f (1 + |\dot{r} - \dot{r}_f|^2)^{-1/2} = \\
 &= -\bar{W}' \dot{r}_f (\bar{W}'^2 - |P|^2)^{1/2} \bar{W}'^{-1} = -\dot{r}_f (\bar{W}'^2 - |P|^2)^{1/2},
 \end{aligned}$$

or

$$(1.95) \quad \dot{r}_f = -\xi_f A (\bar{W}'^2 - |P|^2)^{-1/2},$$

where  $A : M^4 \rightarrow \mathbb{R}^3$  is the related magnetic vector potential generated by the moving external charged particle  $\xi_f$ . Equations (1.92) can be easily rewritten with respect to the laboratory reference frame  $\mathcal{K}$  in the form

$$(1.96) \quad dr/dt = u, \quad dp/dt = \xi E + \xi u \times B - \nabla \langle \xi A, u - u_f \rangle,$$

which coincide with the result (1.75).

Whence, we see that the Hamiltonian function (1.93) satisfies the energy conservation conditions

$$(1.97) \quad dH/dt = 0 = dH/d\tau,$$

for all  $\tau, t \in \mathbb{R}$ , and the suitable energy expression, owing to (1.62), is

$$(1.98) \quad \mathcal{E} = (\bar{W}^2 - |\xi A|^2 - |P|^2)^{1/2} + \langle \xi A, P \rangle (\bar{W}^2 - |\xi A|^2 - |P|^2)^{-1/2},$$

where the generalized momentum  $P = p + \xi A$ . The result (1.98) differs essentially from that obtained in [97], which is strongly based on the Einsteinian Lagrangian for a moving charged point particle  $\xi$  in the external electromagnetic fields, generated by a charged point test particle  $\xi_f$ , moving with velocity  $u_f \in T(\mathbb{R}^3)$  with respect to a laboratory reference frame  $\mathcal{K}$ . Thus, we obtained the following proposition,

**Proposition 1.9.** *The alternative classical relativistic electrodynamic model (1.96), which is intrinsically compatible with the classical Maxwell equations (1.6), allows the Hamiltonian formulation (1.92) with respect to the rest reference frame  $\mathcal{K}_\tau$ , where the Hamiltonian function is given by expression (1.93).*

The inference above is a natural candidate for experimental validation of our vacuum field theory. It is strongly motivated by the following remark.

*Remark 1.10.* It is necessary to mention here that the Lorentz force expression (1.96) uses the particle momentum  $p = mu$ , where the dynamical "mass"  $m := -\bar{W}$  satisfies condition (1.98). The latter gives rise to the following crucial relationship between the particle energy  $\mathcal{E}_0$  and its rest mass  $m_0 = -\bar{W}_0$  (for the velocity  $u = 0$  at the initial time moment  $t = 0$ ):

$$(1.99) \quad \mathcal{E}_0 = m_0 \frac{(1 - |\xi A_0/m_0|^2)}{(1 - 2|\xi A_0/m_0|^2)^{1/2}},$$

or, equivalently, at the condition  $|\xi A_0/m_0|^2 < 1/2$

$$(1.100) \quad m_0 = \mathcal{E}_0 \left( \frac{1}{2} + |\xi A_0/\mathcal{E}_0|^2 \pm \frac{1}{2} \sqrt{1 - 4|\xi A_0/\mathcal{E}_0|^2} \right)^{1/2},$$

where  $A_0 := A|_{t=0} \in \mathbb{E}^3$ , which strongly differs from the classical expression  $m_0 = \mathcal{E}_0 - \xi\varphi_0$ , following from (1.45) and is not depending a priori on the external potential energy  $\xi\varphi_0$ . As the quantity  $|\xi A_0/\mathcal{E}_0| \rightarrow 0$ , the following asymptotical mass values follow from (1.100):

$$(1.101) \quad m_0^{(+)} \simeq \mathcal{E}_0, \quad m_0^{(-)} \simeq \pm\sqrt{2}|\xi A_0|.$$

The first mass value  $m_0^{(+)} \simeq \mathcal{E}_0$  is physically correct, giving rise to the bounded charged particle energy  $\mathcal{E}_0$ , but the second mass value  $m_0^{(-)} \simeq \pm\sqrt{2}|\xi A_0|$  is not physical, as it gives rise to the vanishing denominator  $(1-2|\xi A_0/m_0^{(-)}|^2)^{1/2} \simeq 0$  in (1.99), being equivalent to the unboundedness of the charged particle energy  $\mathcal{E}_0$ .

To make this difference more clear, we now will analyze the dual classical Lorentz force (1.85) from the Hamiltonian point of view, based on the Lagrangian function (1.83). Thus, we can easily obtain that the corresponding Hamiltonian function

$$(1.102) \quad \begin{aligned} H := \langle P, \dot{r} \rangle - \mathcal{L} &= \langle P, \dot{r} \rangle + \bar{W}(1 + |\dot{r}|^2)^{1/2} - \xi \langle A, \dot{r} \rangle = \\ &= \langle P - \xi A, \dot{r} \rangle + \bar{W}(1 + |\dot{r}|^2)^{1/2} = \\ &= -|p|^2 \bar{W}^{-1} (1 - |p|^2/\bar{W}^2)^{-1/2} + \bar{W}(1 - |p|^2/\bar{W}^2)^{-1/2} = \\ &= -(\bar{W}^2 - |p|^2)(\bar{W}^2 - |p|^2)^{-1/2} = -(\bar{W}^2 - |p|^2)^{1/2}. \end{aligned}$$

Since  $p = P - \xi A$ , expression (1.102) assumes the final "no interaction" [97, 116, 96, 131] form

$$(1.103) \quad H = -(\bar{W}^2 - |P - \xi A|^2)^{1/2},$$

which is conserved with respect to the evolution equations (1.81) and (1.82), that is

$$(1.104) \quad dH/dt = 0 = dH/d\tau$$

for all  $\tau, t \in \mathbb{R}$ . These equations are equivalent to the following Hamiltonian system

$$(1.105) \quad \begin{aligned} \dot{r} &= \partial H / \partial P = (P - \xi A)(\bar{W}^2 - |P - \xi A|^2)^{-1/2}, \\ \dot{P} &= -\partial H / \partial r = (\bar{W} \nabla \bar{W} - \nabla \langle \xi A, (P - \xi A) \rangle)(\bar{W}^2 - |P - \xi A|^2)^{-1/2}, \end{aligned}$$

as one can readily check by direct calculations. Actually, the first equation

$$(1.106) \quad \begin{aligned} \dot{r} &= (P - \xi A)(\bar{W}^2 - |P - \xi A|^2)^{-1/2} = p(\bar{W}^2 - |p|^2)^{-1/2} = \\ &= mu(\bar{W}^2 - |p|^2)^{-1/2} = -\bar{W}u(\bar{W}^2 - |p|^2)^{-1/2} = u(1 - |u|^2)^{-1/2}, \end{aligned}$$

holds, owing to the condition  $d\tau = dt(1 - |u|^2)^{1/2}$  and definitions  $p := mu$ ,  $m = -\bar{W}$ , postulated from the very beginning. Similarly we obtain that

$$(1.107) \quad \begin{aligned} \dot{P} &= -\nabla \bar{W}(1 - |p|^2/\bar{W}^2)^{-1/2} + \nabla \langle \xi A, u \rangle (1 - |p|^2/\bar{W}^2)^{-1/2} = \\ &= -\nabla \bar{W}(1 - |u|^2)^{-1/2} + \nabla \langle \xi A, u \rangle (1 - |u|^2)^{-1/2}, \end{aligned}$$

or equivalently, the dual Lorentz dynamical expression

$$(1.108) \quad dp/dt = \xi E + \xi u \times B,$$

exactly coinciding with that of equation (1.85). This result can be reformulated as the next proposition.

**Proposition 1.11.** *The dual to the classical relativistic electrodynamic model (1.85) allows the canonical Hamiltonian formulation (1.105) with respect to the rest reference frame  $\mathcal{K}_\tau$ , where the Hamiltonian function is given by expression (1.103). Moreover, this formulation circumvents the "mass-potential" energy controversy attached to the classical electrodynamic model, based the classical action functional (1.40).*

The classical Lorentz force expression (1.108) and the related conserved energy relationship

$$(1.109) \quad \mathcal{E} = (\bar{W}^2 - |P - \xi A|^2)^{1/2}$$

are characterized by the following remark.

*Remark 1.12.* If we make use of the modified relativistic Lorentz force expression (1.108) as an alternative to the classical one of (1.49), the corresponding charged particle energy relationship (1.109) gives rise to a different energy expression (for the velocity  $u = 0$  at the initial time moment  $t = 0$ ). Namely, one naturally obtains the physically reasonable Einsteinian mass-energy relationship  $\mathcal{E}_0 = m_0$  instead of the senseless classical expression  $\mathcal{E}_0 = m_0 + \xi\varphi_0$ , following from (1.45), where  $\varphi_0 := \varphi|_{t=0}$  and where the mass parameter  $m_0$  is a constant parameter not depending on the external electromagnetic field.

**1.4. Comments.** All of dynamical field equations discussed above are canonical Hamiltonian systems with respect to the corresponding physically proper rest reference frames  $\mathcal{K}_\tau$ , parameterized by the Euclidean coordinates  $(\tau, r) \in \mathbb{E}^4$ . Upon passing to the basic laboratory reference frame  $\mathcal{K}$ , naturally parameterized by the Minkowski coordinates  $(t, r) \in M^4$ , the related Hamiltonian structure is lost, giving rise to a suitably altered interpretation of the real particle motion. Namely, as it was demonstrated above, a least action principle for a charged point particle dynamics makes sense only with respect to the proper rest reference frame  $\mathcal{K}_\tau$  as, otherwise, it becomes completely senseless with respect to all other laboratory reference frames. As for the Hamiltonian expressions (1.88), (1.93) and (1.103), one observes that they all depend strongly on the vacuum potential field function  $\bar{W} : M^4 \rightarrow \mathbb{R}$ , thereby avoiding the mass problem related with the well known classical energy expression and pointed out by L. Brillouin [33].

Some comments are also on order concerning the classical relativity principle and a way of its application to real physical phenomena. We have obtained our results with using the standard Lorentz transformations of reference frames - relying only on the natural notion of the rest reference frame  $\mathcal{K}_\tau$  and its suitable parametrization with respect to any other laboratory reference frame  $\mathcal{K}$ . It seems physically reasonable that the true state changing of a moving charged particle  $\xi$  is exactly realized only with respect to its proper rest reference system  $\mathcal{K}_\tau$ .

Thus, the only remaining question would be about the physical justification of the corresponding relationship between time parameters of the corresponding laboratory and rest reference frames. The relationship between these reference frames that we have used through is simply expressed as

$$(1.110) \quad d\tau = dt(1 - |u|^2)^{1/2},$$

where  $u := dr/dt \in \mathbb{E}^3$  is the velocity vector with which the rest reference frame  $\mathcal{K}_\tau$  moves with respect to other arbitrarily chosen reference frame  $\mathcal{K}$ . Expression (1.110) implies, in particular, that

$$(1.111) \quad dt^2 - |dr|^2 = d\tau^2,$$

which is evidently identical to the classical infinitesimal Lorentz invariant. This is not a coincidence, since all our dynamical vacuum field equations were derived in turn [125, 126] from the governing equations of the vacuum potential field function  $W : M^4 \rightarrow \mathbb{R}$  in the form

$$(1.112) \quad \partial^2 W / \partial t^2 - \nabla^2 W = \rho, \quad \partial W / \partial t + \langle \nabla, vW \rangle = 0, \quad \partial \rho / \partial t + \langle \nabla, v\rho \rangle = 0,$$

which is *a priori* Lorentz invariant. Here  $\rho \in \mathbb{R}$  is the charge density and  $v := dr/dt$  the associated local velocity of the vacuum field potential evolution. Consequently, the dynamical infinitesimal Lorentz invariant (1.111) reflects this intrinsic structure of equations (1.112). If it is rewritten in the following slightly nonstandard Euclidean form:

$$(1.113) \quad dt^2 = d\tau^2 + |dr|^2$$

it gives rise to a completely different relationship between the reference frames  $\mathcal{K}$  and  $\mathcal{K}_\tau$ , namely

$$(1.114) \quad dt = d\tau(1 + |\dot{r}|^2)^{1/2},$$

where  $\dot{r} := dr/d\tau$  is the related particle velocity with respect to the rest reference system  $\mathcal{K}_\tau$ . Thus, we observe that all our Lagrangian analysis is strongly related to the functional expressions written in these "Euclidean" space-time coordinates and with respect to which the least action principle was applied. So we see that there are two alternatives - the first one is to apply the least action principle to the corresponding Lagrangian functions, expressed in the Minkowski space-time variables with respect to an arbitrarily chosen reference frame  $\mathcal{K}$ , and the second one is to apply the least action principle to the corresponding Lagrangian functions expressed in the Euclidean space-time variables with respect to the rest reference frame  $\mathcal{K}_\tau$ . But, as it was demonstrated above, the second alternative appeared to be physically reasonable in contrast to the first one, which gives rise to different physically senseless controversies.



All that above entails also a next slightly amusing but thought-provoking observation: "It follows that all of the results of classical special relativity related with the electrodynamics of charged point particles can be obtained (but not in a one-to-one correspondence) using our new physically reasonable definitions of the dynamical particle mass and the physically motivated least action principle, calculated with respect to the related Euclidean space-time variables specifying the rest reference frame  $\mathcal{K}_\tau$ .

### 1.5. The quantization of electrodynamics models within the vacuum field theory no-geometry approach.

1.5.1. *The problem setting.* Recently [125, 126], we devised a new regular no-geometry approach to deriving the electrodynamics of a moving charged point particle  $q$  in an external electromagnetic field from first principles. This approach has, in part, reconciled the mass-energy controversy [33] in classical relativistic electrodynamics. Using the vacuum field theory approach proposed in [125, 126, 136], we reanalyzed this problem above both from the Lagrangian and Hamiltonian perspective and derived key expressions for the corresponding energy functions and Lorentz type forces acting on a moving charge point particle  $\xi$ .

Since all of our electrodynamics models were represented here in canonical Hamiltonian form, they are well suited to the application of Dirac quantization [40, 28, 29] and the corresponding derivation of related Schrödinger type evolution equations. We describe these procedures in this section.

1.5.2. *Free point particle electrodynamics model and its quantization.* The charged point particle electrodynamics models, discussed in detail in Sections 2 and 3, were also considered in [126] from the dynamical point of view, where a Dirac quantization of the corresponding conserved energy expressions was attempted. However, from the canonical point of view, the true quantization procedure should be based on the relevant canonical Hamiltonian formulation of the models given in (1.91), (1.92) and (1.105).

In particular, consider a free charged point particle electrodynamics model characterized by (1.91) and having the Hamiltonian equations

$$(1.115) \quad \begin{aligned} dr/d\tau &:= \partial H/\partial p = -p(\bar{W}^2 - |p|^2)^{-1/2}, \\ dp/d\tau &:= -\partial H/\partial r = -\bar{W}\nabla\bar{W}(\bar{W}^2 - |p|^2)^{-1/2}, \end{aligned}$$

where  $\bar{W} : M^4 \rightarrow \mathbb{R}$  defined in the preceding sections is the corresponding vacuum field potential characterizing medium field structure,  $(r, p) \in T^*(\mathbb{E}^3) \simeq \mathbb{E}^3 \times \mathbb{E}^3$  are the standard canonical coordinate-momentum variables on the cotangent space  $T^*(\mathbb{E}^3)$ ,  $\tau \in \mathbb{R}$  is the proper rest reference frame  $\mathcal{K}_\tau$  time parameter of the moving particle, and  $H : T^*(\mathbb{E}^3) \rightarrow \mathbb{R}$  is the Hamiltonian function

$$(1.116) \quad H := -(\bar{W}^2 - p^2)^{1/2},$$

expressed here and hereafter in light speed units. The rest reference frame  $\mathcal{K}_\tau$ , parameterized by variables  $(\tau, r) \in \mathbb{E}^4$ , is related to any other reference frame  $\mathcal{K}$  in which our charged point particle  $\xi$  moves with velocity vector  $u \in \mathbb{E}^3$ . The frame  $\mathcal{K}$  is parameterized by variables  $(t, r) \in M^4$  via the Euclidean infinitesimal relationship

$$(1.117) \quad dt^2 = d\tau^2 + |dr|^2,$$

which is equivalent to the Minkowski infinitesimal relationship

$$(1.118) \quad d\tau^2 = dt^2 - |dr|^2.$$

The Hamiltonian function (1.116) clearly satisfies the energy conservation conditions

$$(1.119) \quad dH/dt = 0 = dH/d\tau$$

for all  $t, \tau \in \mathbb{R}$ . This means that the suitable energy

$$(1.120) \quad \mathcal{E} = (\bar{W}^2 - |p|^2)^{1/2}$$

can be treated by means of the Dirac quantization scheme [40, 42, 94] to obtain, as  $\hbar/c \rightarrow 0$ , (or the light speed  $c \rightarrow \infty$ ) the governing Schrödinger type dynamical equation. To do this following

the approach in [125, 126], we need to make canonical operator replacements  $\mathcal{E} \rightarrow \hat{\mathcal{E}} := -\frac{\hbar}{i} \frac{\partial}{\partial \tau}$ ,  $p \rightarrow \hat{p} := \frac{\hbar}{i} \nabla$ , as  $\hbar \rightarrow 0$ , in the following energy expression:

$$(1.121) \quad \mathcal{E}^2 := (\hat{\mathcal{E}}\psi, \hat{\mathcal{E}}\psi) = (\psi, \hat{\mathcal{E}}^2\psi) = (\psi, \hat{H}^+ \hat{H}\psi),$$

where  $(\cdot, \cdot)$  is the standard  $L_2$  - inner product. It follows from (1.120) that

$$(1.122) \quad \hat{\mathcal{E}}^2 = \bar{W}^2 - |\hat{p}|^2 = \hat{H}^+ \hat{H}$$

is a suitable operator factorization in the Hilbert space  $\mathcal{H} := L_2(\mathbb{R}^3; \mathbb{C})$  and  $\psi \in \mathcal{H}$  is the corresponding normalized quantum vector state. Since the following elementary identity

$$(1.123) \quad \bar{W}^2 - |\hat{p}|^2 = \bar{W}(1 - \bar{W}^{-1}|\hat{p}|^2\bar{W}^{-1})^{1/2}(1 - \bar{W}^{-1}|\hat{p}|^2\bar{W}^{-1})^{1/2}\bar{W}$$

holds, we can use (1.122) and (1.123) to define the operator

$$(1.124) \quad \hat{H} := (1 - \bar{W}^{-1}|\hat{p}|^2\bar{W}^{-1})^{1/2}\bar{W}.$$

Having calculated the operator expression (1.124) as  $\hbar \rightarrow 0$  up to operator accuracy  $O(\hbar^4)$ , it is easy to see that

$$(1.125) \quad \hat{H} = \frac{|\hat{p}|^2}{2m(u)} + \bar{W} := -\frac{\hbar^2}{2m(u)}\Delta + \bar{W},$$

where we have taken into account the dynamical mass definition  $m(u) := -\bar{W}$  (in the light speed units). Consequently, using (1.121) and (1.125), we obtain up to operator accuracy  $O(\hbar^4)$  the following Schrödinger type evolution equation

$$(1.126) \quad i\hbar \frac{\partial \psi}{\partial \tau} := \hat{\mathcal{E}}\psi = \hat{H}\psi = -\frac{\hbar^2}{2m(u)}\Delta\psi + \bar{W}\psi$$

with respect to the rest reference frame  $\mathcal{K}_r$  evolution parameter  $\tau \in \mathbb{R}$ . For a related evolution parameter  $t \in \mathbb{R}$  parameterizing a reference frame  $\mathcal{K}$ , the equation (1.126) takes the form

$$(1.127) \quad i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2 m_0}{2m(u)^2}\Delta\psi - m_0\psi.$$

Here we used the fact that it follows from (1.120) that the classical inertial mass relationship

$$(1.128) \quad m(u) = m_0(1 - |u|^2)^{-1/2}$$

holds, where  $m_0 \in \mathbb{R}_+$  is the corresponding rest mass of our point particle  $\xi$ .

The linear Schrödinger equation (1.127) for the case  $\hbar/c \rightarrow 0$  actually coincides with the well-known expression [97, 40, 57] from classical quantum mechanics.

**1.5.3. Classical charged point particle electrodynamics model and its quantization.** We start here from the first vacuum field theory reformulation of the classical charged point particle electrodynamics (introduced in Section 3) and based on the conserved Hamiltonian function (1.103)

$$(1.129) \quad H := -(\bar{W}^2 - |P - \xi A|^2)^{1/2},$$

where  $q \in \mathbb{R}$  is the particle charge,  $(\bar{W}, A) \in \mathbb{R} \times \mathbb{E}^3$  is the corresponding representation of the electromagnetic field potentials and  $P \in \mathbb{E}^3$  is the common generalized particle-field momentum

$$(1.130) \quad P := p + \xi A, \quad p := mu,$$

which satisfies the classical Lorentz force equation. Here  $m := -\bar{W}$  is the observable dynamical mass of our charged particle, and  $u \in \mathbb{E}^3$  is its velocity vector with respect to a chosen reference frame  $\mathcal{K}$ , all expressed in light speed units.

Our electrodynamics based on (1.129) is canonically Hamiltonian, so the Dirac quantization scheme

$$(1.131) \quad P \rightarrow \hat{P} := \frac{\hbar}{i} \nabla, \quad \mathcal{E} \rightarrow \hat{\mathcal{E}} := -\frac{\hbar}{i} \frac{\partial}{\partial \tau}$$

should be applied to the energy expression

$$(1.132) \quad \mathcal{E} := (\bar{W}^2 - |P - \xi A|^2)^{1/2},$$

following from the conservation conditions

$$(1.133) \quad dH/dt = 0 = dH/d\tau,$$

satisfied for all  $\tau, t \in \mathbb{R}$ .

Proceeding as above, we can factorize the operator  $\hat{\mathcal{E}}^2$  as

$$\begin{aligned} \bar{W}^2 - |\hat{P} - \xi A|^2 &= \bar{W}(1 - \bar{W}^{-1}|\hat{P} - \xi A|^2\bar{W})^{1/2} \times \\ &\times (1 - \bar{W}^{-1}|\hat{P} - \xi A|^2\bar{W}^{-1})^{1/2}\bar{W} := \hat{H}^+ \hat{H}, \end{aligned}$$

where (as  $\hbar/c \rightarrow 0$ ,  $\hbar c = \text{const}$ )

$$(1.134) \quad \hat{H} := \frac{1}{2m(u)} \left| \frac{\hbar}{i} \nabla - \xi A \right|^2 + \bar{W}$$

up to operator accuracy  $O(\hbar^4)$ . Hence, the related Schrödinger type evolution equation in the Hilbert space  $\mathcal{H} = L_2(\mathbb{R}^3; \mathbb{C})$  is

$$(1.135) \quad i\hbar \frac{\partial \psi}{\partial \tau} := \hat{\mathcal{E}}\psi = \hat{H}\psi = \frac{1}{2m(u)} \left| \frac{\hbar}{i} \nabla - \xi A \right|^2 \psi + \bar{W}\psi$$

with respect to the rest reference frame  $\mathcal{K}_r$  evolution parameter  $\tau \in \mathbb{R}$ , and corresponding Schrödinger type evolution equation with respect to the evolution parameter  $t \in \mathbb{R}$  takes the form

$$(1.136) \quad i\hbar \frac{\partial \psi}{\partial t} = \frac{m_0}{2m(u)^2} \left| \frac{\hbar}{i} \nabla - \xi A \right|^2 \psi - m_0 \psi.$$

The Schrödinger equation (1.135) (as  $\hbar/c \rightarrow 0$ ) coincides [98, 40] with the classical quantum mechanics version.

**1.5.4. Modified charged point particle electrodynamics model and its quantization.** From the canonical viewpoint, we now turn to the true quantization procedure for the electrodynamics model, characterized by (1.69) and having the Hamiltonian function (1.93)

$$(1.137) \quad H := -(\bar{W}^2 - |P|^2)^{1/2} - \xi \langle A, P \rangle (\bar{W}^2 - |P|^2)^{-1/2}.$$

Accordingly the suitable energy function is

$$(1.138) \quad \mathcal{E} := (\bar{W}^2 - |P|^2)^{1/2} + \xi \langle A, P \rangle (\bar{W}^2 - |P|^2)^{-1/2},$$

where, as before,

$$(1.139) \quad P := p + \xi A, \quad p := mu, \quad m := -\bar{W},$$

is a conserved quantity for (1.69), which we will canonically quantize via the Dirac procedure (1.131). Toward this end, let us consider the quantum condition

$$(1.140) \quad \mathcal{E}^2 := (\hat{\mathcal{E}}\psi, \hat{\mathcal{E}}\psi) = (\psi, \hat{\mathcal{E}}^2\psi), \quad (\psi, \psi) := 1,$$

where, by definition,  $\hat{\mathcal{E}} := -\frac{\hbar}{i} \frac{\partial}{\partial t}$  and  $\psi \in \mathcal{H} = L_2(\mathbb{R}^3; \mathbb{C})$  is a respectively normalized quantum state vector. Making now use of the energy function (1.138), one readily computes that

$$(1.141) \quad \mathcal{E}^2 = \bar{W}^2 - |P - qA|^2 + \xi^2 \langle A, A \rangle + \xi^2 \langle A, P \rangle (\bar{W}^2 - |P|^2)^{-1} \langle P, A \rangle,$$

which transforms by the canonical Dirac type quantization  $P \rightarrow \hat{P} := \frac{\hbar}{i} \nabla$  into the symmetrized operator expression

$$(1.142) \quad \hat{\mathcal{E}}^2 = \bar{W}^2 - |\hat{P} - \xi A|^2 + \xi^2 \langle A, A \rangle + \xi^2 \langle A, \hat{P} \rangle (\bar{W}^2 - |\hat{P}|^2)^{-1} \langle \hat{P}, A \rangle.$$

Factorizing the operator (1.142) in the form  $\hat{\mathcal{E}}^2 = \hat{H}^+ \hat{H}$ , and retaining only terms up to  $O(\hbar^4)$  (as  $\hbar/c \rightarrow 0$ ), we compute that

$$(1.143) \quad \begin{aligned} \hat{H} &:= \frac{1}{2m(u)} \left| \frac{\hbar}{i} \nabla - \xi A \right|^2 - \frac{\xi^2}{2m(u)} \langle A, A \rangle - \\ &- \frac{\xi^2}{2m^3(u)} \langle A, \frac{\hbar}{i} \nabla \rangle \langle \frac{\hbar}{i} \nabla, A \rangle, \end{aligned}$$

where, as before,  $m(u) = -\bar{W}$  in light speed units. Thus, owing to (1.140) and (1.143), the resulting Schrödinger evolution equation is

$$(1.144) \quad \begin{aligned} i\hbar \frac{\partial \psi}{\partial \tau} &:= \hat{H}\psi = \frac{1}{2m(u)} \left| \frac{\hbar}{i} \nabla - qA \right|^2 \psi - \\ &- \frac{q^2}{2m(u)} \langle A, A \rangle \psi - \frac{\xi^2}{2m^3(u)} \langle A, \frac{\hbar}{i} \nabla \rangle \langle \frac{\hbar}{i} \nabla, A \rangle \psi \end{aligned}$$

with respect to the rest reference frame proper evolution parameter  $\tau \in \mathbb{R}$ . The latter can be rewritten in the equivalent form as

$$(1.145) \quad \begin{aligned} i\hbar \frac{\partial \psi}{\partial \tau} &= -\frac{\hbar^2}{2m(u)} \Delta \psi - \frac{1}{2m(u)} < [\frac{\hbar}{i} \nabla, qA]_+ > \psi - \\ -\frac{\xi^2}{2m^3(u)} &< A, \frac{\hbar}{i} \nabla > < \frac{\hbar}{i} \nabla, A > \psi, \end{aligned}$$

where  $[\cdot, \cdot]_+$  means the formal anti-commutator of operators. Similarly one also obtains the related Schrödinger equation with respect to the time parameter  $t \in \mathbb{R}$ , which we shall not dwell upon here. The result (1.144) only slightly differs from the classical Schrödinger evolution equation (1.135). Simultaneously, its form (1.145) almost completely coincides with the classical ones from [98, 116, 40] modulo the evolution considered with respect to the rest reference time parameter  $\tau \in \mathbb{R}$ . This suggests that we must more thoroughly reexamine the physical motivation of the principles underlying the classical electrodynamic models, described by the Hamiltonian functions (1.129) and (1.137), giving rise to different Lorentz type force expressions. A more deeply considered and extended analysis of this matter is forthcoming in a paper now in preparation.

**1.6. Comments.** All of dynamical field equations discussed above are canonical Hamiltonian systems with respect to the corresponding proper rest reference frames  $\mathcal{K}_r$ , parameterized by suitable time parameters  $\tau \in \mathbb{R}$ . Upon passing to the basic laboratory reference frame  $\mathcal{K}$  with the time parameter  $t \in \mathbb{R}$ , naturally the related Hamiltonian structure is lost, giving rise to a new interpretation of the real particle motion. Namely, one that has an absolute sense only with respect to the proper rest reference system, and otherwise being completely relative with respect to all other reference frames. As for the Hamiltonian expressions (1.88), (1.93) and (1.103), one observes that they all depend strongly on the vacuum potential field function  $\bar{W} : M^4 \rightarrow \mathbb{R}$ , thereby avoiding the mass problem of the classical energy expression pointed out by L. Brillouin [33]. It should be noted that the canonical Dirac quantization procedure can be applied only to the corresponding dynamical field systems considered with respect to their proper rest reference frames.

An additional remark concerning the quantization procedure of the proposed electrodynamics models is in order: If the dynamical vacuum field equations are expressed in canonical Hamiltonian form, as we have done in this paper, only straightforward technical details are required to quantize the equations and obtain the corresponding Schrödinger evolution equations in suitable Hilbert spaces of quantum states. There is another striking implication from our approach: the Einsteinian equivalence principle [97, 116, 57, 56, 88] is rendered superfluous for our vacuum field theory of electromagnetism and gravity.

Using the canonical Hamiltonian formalism devised here for the alternative charged point particle electrodynamics models, we found it rather easy to treat the Dirac quantization. The results obtained compared favorably with classical quantization, but it must be admitted that we still have not given a compelling physical motivation for our new models. This is something that we plan to revisit in future investigations. Another important aspect of our vacuum field theory no-geometry (geometry-free) approach to combining the electrodynamics with the gravity, is the manner in which it singles out the decisive role of the rest reference frame  $\mathcal{K}_r$ . More precisely, all of our electrodynamics models allow both the Lagrangian and Hamiltonian formulations with respect to the rest reference system evolution parameter  $\tau \in \mathbb{R}$ , which are well suited to canonical quantization. The physical nature of this fact remains as yet not quite clear. In fact, as far as we know [116, 97, 88, 100, 101], there is no physically reasonable explanation of this decisive role of the rest reference system, except for that given by R. Feynman who argued in [57] that the relativistic expression for the classical Lorentz force (1.49) has physical sense only with respect to the rest reference frame variables  $(\tau, r) \in \mathbb{E}^4$ . In future research we plan to analyze the quantization scheme in more detail and begin work on formulating a vacuum quantum field theory of infinitely many particle systems.

## 2. THE MODIFIED LORENTZ FORCE AND CHARGE RADIATION PROBLEM WITHIN THE VACUUM FIELD THEORY APPROACH

**2.1. Introductory setting.** The Maxwell equations, being a one of physical fundamental theories nowadays allow, as is well known two main forms of representations: either by means of the

electric and magnetic fields or by means of the electric and magnetic potentials. The latter were mainly considered as a mathematically motivated representation useful for different applications but having no physical significance.

That the situation is not so simple and the evidence that the magnetic potential demonstrates the physical properties was doubtless, the physics community understood when Y. Aharonov and D. Bohm [2] formulated their "paradox" concerning the measurement of magnetic field outside a separated region where it is completely vanishing. Later such similar effects were also revealed in the superconductivity theory of Josephson media. As the existence of any electromagnetic field in the ambient space can be tested only owing to its interaction with electric charges, their dynamical behavior, being of great importance, was deeply studied by M. Faraday, A. Ampere and H. Lorentz subject to its classical second Newton law form. Namely, the classical Lorentz force

$$(2.1) \quad dp/dt = \xi E + \xi \frac{u}{c} \times B$$

was derived, where  $E$  and  $B \in \mathbb{E}^3$  are, respectively, electric and magnetic fields, acting on a point charged particle  $\xi \in \mathbb{R}$ , possessing the momentum  $p = mu$ ,  $m \in \mathbb{R}_+$  is the observed particle mass and  $u \in T(\mathbb{R}^3)$  is its velocity, measured with respect to a suitably chosen laboratory reference frame  $\mathcal{K}$ .

That the Lorentz force (2.1) is not a completely satisfactory expression was well known still for Lorentz himself, as the nonuniform Maxwell equations describe also the electromagnetic fields, radiated by any accelerated charged particle, easily seen from well-known expressions for the Lienard-Wiechert electromagnetic four-potential  $(\varphi, A) : M^4 \rightarrow T^*(M^4)$ , related to the electromagnetic fields by means of the well known [97, 82, 39] relationships

$$(2.2) \quad E := -\nabla\varphi - \frac{1}{c} \frac{\partial A}{\partial t}, B := \nabla \times A.$$

This fact had inspired many physicists to "improve" the classical Lorentz force expression (2.1) and its modification was then suggested by G.A. Shott [141] and later by M. Abraham and P.A.M. Dirac (see [82, 39]), who found that the so called classical "radiation reaction" force, owing to the self-interaction of a charged particle with charge  $\xi \in \mathbb{R}$ , equals

$$(2.3) \quad \frac{dp}{dt} = \xi E + \xi \frac{u}{c} \times B + \frac{2\xi^2}{3c^3} \frac{d^2u}{dt^2}.$$

The additional self-reaction force expression

$$(2.4) \quad F_r := \frac{2\xi^2}{3c^3} \frac{d^2u}{dt^2},$$

depending on the particle acceleration had entailed right away many questions concerning its physical meaning, since for instance, a uniformly accelerated charged particle, owing to the expression (2.3), does feel no radiation reaction, contradicting the fact that any accelerated charged particle always radiates electromagnetic waves. This "paradox" was a challenging problem during the XX century [141, 41, 82, 30, 117, 82] and still remains to be not explained completely [137, 106, 105] up to present days. As there exist different approaches to explanation this reaction radiation phenomenon, we mention here only such most popular ones as the Wheeler-Feynman's [156] "absorber radiation" theory, based on a very sophisticated elaboration of the retarded and advanced solutions to the nonuniform Maxwell equations, the vacuum Casimir effect approach devised in [111, 144], and the construction of Teitelbom [147] which extensively exploits the intrinsic structure of the electromagnetic energy tensor subject to the advanced and retarded solutions to the nonuniform Maxwell equations.

It is also worth of mentioning here very nontrivial development of the Teitelbom's theory devised recently in [90, 143] and applied to the non-abelian Yang-Mills equations, naturally generalizing the classical Maxwell equations. Nonetheless, all of these explanations do not prove to be satisfactory from the modern physics of view. Taking this state of art into account we will reanalyze once more the structure of the "radiative" Lorentz type force (2.3) from the vacuum field theory approach of the Section 3.1 and obtain that this force allows some natural slight modification.

**2.2. The radiation reaction force: the vacuum-field theory approach.** In the Section to proceed below, we will develop further our vacuum field theory approach devised before in [19, 127, 125, 126] to the electromagnetic Maxwell and Lorentz electron theories and will show

that it is in complete agreement with the classical results and even more, it allows some nontrivial generalizations, which may have some nontrivial physical applications. It will be also shown that the closely related electron mass problem can be satisfactorily explained within the devised vacuum field theory approach and the spatial electron structure assumption.

The modified Lorentz force, acting on a particle of charge  $\xi \in \mathbb{R}$  and exerted by a moving with velocity  $u_f \in T(\mathbb{R}^3)$  charged particle  $\xi_f \in \mathbb{R}$ , was derived in Section 3.1 and equals

$$(2.5) \quad dp/dt := F_s = \xi E + \xi \frac{u}{c} \times B - \nabla < \xi A/c, u - u_f >,$$

where  $(\varphi, A) \in T^*(M^4)$  is the external electromagnetic potential calculated with respect to a fixed laboratory reference frame  $\mathcal{K}$ . To take into account the self-interaction of this particle we will make use of a spatially distributed charge density  $\rho : M^4 \rightarrow \mathbb{R}$ , satisfying the condition

$$(2.6) \quad \xi = \int_{\mathbb{R}^3} \rho(t, r) d^3 r$$

for all  $t \in \mathbb{R}$  subject to this laboratory reference frame  $\mathcal{K}$  with coordinates  $(t, r) \in M^4$ . Then, owing to 2.5 and the reasonings from Section 3.1, the self-interacting force of this spatially structured charge  $\xi \in \mathbb{R}$  can be expressed with respect to this laboratory reference frame  $\mathcal{K}$  in the following equivalent form:

$$(2.7) \quad dp/dt = -\frac{1}{c} \frac{d}{dt} \left[ \int_{\mathbb{R}^3} d^3 r \rho(t, r) A_s(t, r) \right] - \int_{\mathbb{R}^3} d^3 r \rho(t, r) \nabla \varphi_s(t, r) (1 - |u/c|^2) =$$

where we denoted by

$$(2.8) \quad \varphi_s(t, r) = \int_{\mathbb{R}^3} \frac{\rho(t', r')|_{ret} d^3 r'}{|r - r'|}, \quad A_s(t, r) = \frac{1}{c} \int_{\mathbb{R}^3} \frac{u(t') \rho(t', r')|_{ret} d^3 r'}{|r - r'|},$$

the well-known retarded Lienard-Wiechert potentials, which should be calculated at the retarded time parameter  $t' := t - |r - r'|/c \in \mathbb{R}$ . Taking additionally into account the continuity relationship

$$(2.9) \quad \partial \rho / \partial t + \nabla \cdot J = 0$$

for the spatially distributed charge density  $\rho : M^4 \rightarrow \mathbb{R}$  and current  $J = \rho u : M^4 \rightarrow \mathbb{E}^3$  and the Taylor expansions for retarded potentials (2.8)

$$(2.10) \quad \begin{aligned} \varphi_s(t, r) &= \sum_{n \in \mathbb{Z}_+} \frac{\partial^n}{\partial t^n} \int_{\mathbb{R}^3} \frac{(-|r - r'|)^n}{c^n n!} \frac{\rho(t, r') d^3 r'}{|r - r'|}, \\ A_s(t, r) &= \sum_{n \in \mathbb{Z}_+} \frac{\partial^n}{\partial t^n} \int_{\mathbb{R}^3} \frac{(-|r - r'|)^n}{c^n n!} \frac{J(t, r') d^3 r'}{|r - r'|}, \end{aligned}$$

from (2.7) and (2.10), assuming for brevity the spherical charge distribution, small enough value  $|u|/c \ll 1$  and, respectively, slow acceleration, followed by calculations similar to those of [82, 105], one can obtain that

$$(2.11) \quad \begin{aligned} F_s &= \sum_{n \in \mathbb{Z}_+} \frac{(-1)^{n+1}}{n! c^n} (1 - |u/c|^2) \int_{\mathbb{R}^3} d^3 r \rho(t, r) \int_{\mathbb{R}^3} d^3 r' \frac{\partial^n}{\partial t^n} \rho(t, r') \nabla |r - r'|^{n-1} + \\ &\quad + \frac{d}{dt} \left[ \sum_{n \in \mathbb{Z}_+} \frac{(-1)^{n+1}}{n! c^n} \int_{\mathbb{R}^3} d^3 r \rho(t, r) \frac{|r - r'|^{n-1}}{c^2} \frac{\partial^n}{\partial t^n} J(t, r') \right] = \\ &= \sum_{m \in \mathbb{Z}_+} \frac{(-1)^{m+1}}{(m+2)! c^{m+2}} (1 - |u/c|^2) \int_{\mathbb{R}^3} d^3 r \rho(t, r) \int_{\mathbb{R}^3} d^3 r' \frac{\partial^{m+2}}{\partial t^{m+2}} \rho(t, r') \nabla |r - r'|^{m+1} + \\ &\quad + \frac{d}{dt} \left[ \sum_{m \in \mathbb{Z}_+} \frac{(-1)^{m+1}}{m! c^m} \int_{\mathbb{R}^3} d^3 r \rho(t, r) \int_{\mathbb{R}^3} d^3 r' \frac{\partial^m}{\partial t^m} \left( \frac{|r - r'|^{m+1}}{c^2} J(t, r') \right) \right] = \\ &= \sum_{m \in \mathbb{Z}_+} \frac{(-1)^{m+1}}{m! c^{m+2}} \int_{\mathbb{R}^3} d^3 r \rho(t, r) (1 - |u/c|^2) \int_{\mathbb{R}^3} d^3 r' \frac{\partial^{m+1}}{\partial t^{m+1}} < J(t, r'), \nabla > |r - r'|^{m+1} = \\ &\quad + \frac{d}{dt} \left[ \sum_{m \in \mathbb{Z}_+} \frac{(-1)^{m+1}}{m! c^m} \int_{\mathbb{R}^3} d^3 r \rho(t, r) \int_{\mathbb{R}^3} d^3 r' \frac{|r - r'|^{m+1}}{c^2} \frac{\partial^m}{\partial t^m} (J(t, r')) \right] = \\ &= \sum_{n \in \mathbb{Z}_+} \frac{(-1)^{n+1}}{n! c^{n+2}} (1 - |u|^2) \int_{\mathbb{R}^3} d^3 r \rho(t, r) \int_{\mathbb{R}^3} d^3 r' |r - r'|^{n-1} \frac{\partial^{n+1}}{\partial t^{n+1}} \left[ \frac{\partial \rho(t, r')}{\partial t} \frac{\nabla |r - r'|^{n+1}}{(n+2)(n+1)|r - r'|^{n-1}} + J(t, r') \right] + \\ &\quad + \frac{d}{dt} \left[ \sum_{m \in \mathbb{Z}_+} \frac{(-1)^{m+1}}{m! c^m} \int_{\mathbb{R}^3} d^3 r \rho(t, r) \int_{\mathbb{R}^3} d^3 r' \frac{|r - r'|^{m+1}}{c^2} \frac{\partial^m}{\partial t^m} J(t, r') \right] = \end{aligned}$$

$$\begin{aligned}
&= \sum_{n \in \mathbb{Z}_+} \frac{(-1)^{n+1}}{n!c^{n+2}} (1 - |u|^2) \int_{\mathbb{E}^3} d^3r \rho(t, r) (1 - |u|^2) \int_{\mathbb{E}^3} d^3r' \frac{\partial^{n+1}}{\partial t^{n+1}} [-\langle \nabla, J(t, r') \rangle \frac{|r-r'|^{n-1}(r-r')}{(n+2)}] + \\
&\quad + \frac{d}{dt} [\sum_{m \in \mathbb{Z}_+} \frac{(-1)^{m+1}}{m!c^m} \int_{\mathbb{E}^3} d^3r \rho(t, r) \int_{\mathbb{E}^3} d^3r' \frac{|r-r'|^{m+1}}{c^2} \frac{\partial^m}{\partial t^m} J(t, r')] = \\
&= \sum_{n \in \mathbb{Z}_+} \frac{(-1)^n}{n!c^{n+2}} (1 - |u/c|^2) \int_{\mathbb{E}^3} d^3r \rho(t, r) \int_{\mathbb{E}^3} d^3r' |r-r'|^{n-1} \frac{\partial^{n+1}}{\partial t^{n+1}} \left( \frac{J(t, r')}{n+2} + \frac{n-1}{n+2} \frac{\langle r-r', J(t, r') \rangle}{|r-r'|^2} \right) + \\
&\quad + \frac{d}{dt} [\sum_{m \in \mathbb{Z}_+} \frac{(-1)^{m+1}}{m!c^m} \int_{\mathbb{E}^3} d^3r \rho(t, r) \int_{\mathbb{E}^3} d^3r' \frac{|r-r'|^{m+1}}{c^2} \frac{\partial^m}{\partial t^m} J(t, r')].
\end{aligned}$$

The relationship above can be rewritten, owing to the charge continuity equation (2.9), gives rise to the radiation force expression (2.12)

$$\begin{aligned}
F_s &= \sum_{n \in \mathbb{Z}_+} \frac{(-1)^n}{n!c^{n+2}} (1 - |u/c|^2) \int_{\mathbb{E}^3} d^3r \rho(t, r) \int_{\mathbb{E}^3} d^3r' |r-r'|^{n-1} \frac{\partial^{n+1}}{\partial t^{n+1}} \left( \frac{J(t, r')}{n+2} + \frac{n-1}{n+2} \frac{\langle r-r', J(t, r') \rangle}{|r-r'|^2} \right) + \\
&\quad + \frac{d}{dt} [\sum_{m \in \mathbb{Z}_+} \frac{(-1)^{m+1}}{m!c^m} \int_{\mathbb{E}^3} d^3r \rho(t, r) \int_{\mathbb{E}^3} d^3r' \frac{|r-r'|^{m+1}}{c^2} \frac{\partial^m}{\partial t^m} J(t, r')] = \\
&= \sum_{n \in \mathbb{Z}_+} \frac{(-1)^{n+1}}{n!c^{n+2}} (1 - |u/c|^2) \int_{\mathbb{E}^3} d^3r \rho(t, r) \int_{\mathbb{E}^3} d^3r' |r-r'|^{n-1} \frac{\partial^{n+1}}{\partial t^{n+1}} \left( \frac{J(t, r')}{n+2} + \frac{n-1}{n+2} \frac{|r-r'|^2 J(t, r')}{|r-r'|^2 |u|^2} \right) + \\
&\quad + \frac{d}{dt} [\sum_{m \in \mathbb{Z}_+} \frac{(-1)^{m+1}}{m!c^m} \int_{\mathbb{E}^3} d^3r \rho(t, r) \int_{\mathbb{E}^3} d^3r' \frac{|r-r'|^{m+1}}{c^2} \frac{\partial^m}{\partial t^m} J(t, r')].
\end{aligned}$$

Now, having applied to (2.12) the rotational symmetry property for calculation of the internal integral, one easily obtains that

$$\begin{aligned}
F_s &= \sum_{n \in \mathbb{Z}_+} \frac{(-1)^n}{n!c^{n+2}} (1 - |u/c|^2) \int_{\mathbb{E}^3} d^3r \rho(t, r) \int_{\mathbb{E}^3} d^3r' |r-r'|^{n-1} \frac{\partial^{n+1}}{\partial t^{n+1}} \left( \frac{J(t, r')}{n+2} + \frac{(n-1)J(t, r')}{3(n+2)} \right) + \\
&\quad + \frac{d}{dt} [\sum_{m \in \mathbb{Z}_+} \frac{(-1)^{m+1}}{m!c^m} \int_{\mathbb{E}^3} d^3r \rho(t, r) \int_{\mathbb{E}^3} d^3r' \frac{|r-r'|^{m+1}}{c^2} \frac{\partial^m}{\partial t^m} J(t, r')] = \\
(2.13) \quad &= \sum_{n \in \mathbb{Z}_+} \frac{(-1)^n}{3n!c^{n+2}} (1 - |u/c|^2) \int_{\mathbb{E}^3} d^3r \rho(t, r) \int_{\mathbb{E}^3} d^3r' |r-r'|^{n-1} \frac{\partial^{n+1}}{\partial t^{n+1}} J(t, r') + \\
&\quad + \frac{d}{dt} [\sum_{m \in \mathbb{Z}_+} \frac{(-1)^{m+1}}{m!c^m} \int_{\mathbb{E}^3} d^3r \rho(t, r) \int_{\mathbb{E}^3} d^3r' \frac{|r-r'|^{m+1}}{c^2} \frac{\partial^m}{\partial t^m} J(t, r')] = \\
&= \frac{d}{dt} [\sum_{m \in \mathbb{Z}_+} \frac{(-1)^{m+1}}{m!c^m} \int_{\mathbb{E}^3} d^3r \rho(t, r) \int_{\mathbb{E}^3} d^3r' \frac{|r-r'|^{m+1}}{c^2} \frac{\partial^m}{\partial t^m} [J(t, r') - \frac{1}{3}J(t, r')] - \\
&\quad - \sum_{n \in \mathbb{Z}_+} \frac{(-1)^m}{3m!c^{m+2}} (1 - |u/c|^2) \int_{\mathbb{E}^3} d^3r \frac{\partial \rho(t, r)}{\partial t} \int_{\mathbb{E}^3} d^3r' |r-r'|^{m-1} \frac{\partial^{m+1}}{\partial t^{m+1}} J(t, r') = \\
&= \frac{d}{dt} [\sum_{m \in \mathbb{Z}_+} \frac{(-1)^{m+1}}{m!c^m} \int_{\mathbb{E}^3} d^3r \rho(t, r) \int_{\mathbb{E}^3} d^3r' \frac{|r-r'|^{m+1}}{c^2} \frac{\partial^m}{\partial t^m} [J(t, r') - \frac{1}{3}J(t, r')] - \\
&\quad - \sum_{n \in \mathbb{Z}_+} \frac{(-1)^m}{3m!c^{m+2}} (1 - |u/c|^2) \int_{\mathbb{E}^3} d^3r \frac{\partial \rho(t, r)}{\partial t} \int_{\mathbb{E}^3} d^3r' |r-r'|^{m-1} \frac{\partial^{m+1}}{\partial t^{m+1}} J(t, r') = \\
&= \frac{d}{dt} [\sum_{m \in \mathbb{Z}_+} \frac{(-1)^{m+1}}{m!c^m} \int_{\mathbb{E}^3} d^3r \rho(t, r) \int_{\mathbb{E}^3} d^3r' \frac{|r-r'|^{m+1}}{c^2} \frac{\partial^m}{\partial t^m} [J(t, r') - \frac{1}{3}J(t, r')] + \\
&\quad + \sum_{n \in \mathbb{Z}_+} \frac{(-1)^m}{3m!c^{m+2}} (1 - |u/c|^2) \int_{\mathbb{E}^3} d^3r \langle \nabla, J(t, r') \rangle \int_{\mathbb{E}^3} d^3r' |r-r'|^{m-1} \frac{\partial^{m+1}}{\partial t^{m+1}} J(t, r') = \\
&= \frac{d}{dt} [\sum_{m \in \mathbb{Z}_+} \frac{(-1)^{m+1}}{m!c^m} \int_{\mathbb{E}^3} d^3r \rho(t, r) \int_{\mathbb{E}^3} d^3r' \frac{|r-r'|^{m+1}}{c^2} \frac{\partial^m}{\partial t^m} [J(t, r') - \frac{1}{3}J(t, r')],
\end{aligned}$$

where we took into account [82] that in case of the spherical charge distribution the following equalities

$$\begin{aligned}
 (2.14) \quad & \int_{\mathbb{E}^3} d^3r \int_{\mathbb{E}^3} d^3r' \rho(t, r) \rho(t, r') \frac{|\leq r-r', u(t) \geq|^2}{|r-r'|^2 |u(t)|^2} = \frac{1}{3} e^2, \\
 & \int_{\mathbb{E}^3} d^3r < \nabla, J(t, r') > \int_{\mathbb{E}^3} d^3r' |r - r'|^{m-1} \frac{\partial^{m+1}}{\partial t^{m+1}} J(t, r') = 0, \\
 & \int_{\mathbb{E}^3} d^3r \int_{\mathbb{E}^3} d^3r' \rho(t, r) \rho(t, r') \frac{(r-r')}{|r-r'|^3} = 0
 \end{aligned}$$

hold. Thus, from (2.13) one easily finds up to the  $O(1/c^4)$  accuracy the following radiation reaction force expression:

$$\begin{aligned}
 (2.15) \quad dp/dt &= F_s = -\frac{d}{dt} \left( \frac{4\mathcal{E}_{es}}{3c^2} u(t) \right) - \frac{d}{dt} \left( \frac{2\mathcal{E}_{es}}{3c^2} |u/c|^2 u(t) \right) + \frac{2\xi^2}{3c^3} \frac{d^2u}{dt^2} + O(1/c^4) = \\
 &= -\frac{d}{dt} \left( \frac{4}{3} m_{0,es} \left( 1 + \frac{|u/c|^2}{2} \right) u(t) \right) + \frac{2\xi^2}{3c^3} \frac{d^2u}{dt^2} + O(1/c^4) = \\
 &= -\frac{d}{dt} \left( \frac{4}{3} \frac{m_{0,es} u(t)}{(1 - |u/c|^2)^{1/2}} \right) + \frac{2\xi^2}{3c^3} \frac{d^2u}{dt^2} + O(1/c^4) = \\
 &= -\frac{d}{dt} \left( \frac{4}{3} m_{es} u(t) \right) + \frac{2\xi^2}{3c^3} \frac{d^2u}{dt^2} + O(1/c^4),
 \end{aligned}$$

where we defined, respectively, the electrostatic self-interaction repulsive energy as

$$(2.16) \quad \mathcal{E}_{es} := \frac{1}{2} \int_{\mathbb{E}^3} d^3r \int_{\mathbb{E}^3} d^3r' \frac{\rho(t, r) \rho(t, r')}{|r - r'|},$$

the electromagnetic charged particle rest and inertial masses as

$$(2.17) \quad m_{0,es} := \frac{\mathcal{E}_{es}}{c^2}, \quad m_{es} := \frac{m_{0,es}}{(1 - |u/c|^2)^{1/2}}.$$

Now from (2.5) one obtains that

$$(2.18) \quad \frac{d}{dt} \left[ \left( m_g + \frac{4}{3} m_{es} \right) u \right] = \frac{2\xi^2}{3c^3} \frac{d^2u}{dt^2} + O(1/c^4),$$

where we made use of the inertial mass definition

$$(2.19) \quad m_g := -\bar{W}_g/c^2, \quad \nabla \bar{W}_g \simeq 0,$$

following from the vacuum field theory approach, where the  $m_g \in \mathbb{R}$  is the corresponding gravitational mass of the charged particle  $\xi$ , generated by the vacuum field potential  $\bar{W}_g$ . The corresponding radiation force

$$(2.20) \quad F_r = \frac{2\xi^2}{3c^3} \frac{d^2u}{dt^2} + O(1/c^4),$$

coinciding exactly with the classical Abraham-Lorentz-Dirac results. From (2.18) one follows that the observable physical charged particle mass  $m_{ph} \simeq m_g + \frac{4}{3} m_{es}$  consists of two impacts: the electromagnetic and gravitational components, giving rise to the final force expression

$$(2.21) \quad \frac{d}{dt} (m_{ph} u) = \frac{2\xi^2}{3c^3} \frac{d^2u}{dt^2} + O(1/c^4).$$

It means, in particular, that the real physically observed "inertial" mass  $m_{ph}$  of a real electron strongly depends on the external physical interaction with the ambient vacuum medium, as it was recently demonstrated within completely different approaches in [144, 111], based on the vacuum Casimir effect considerations. Moreover, the assumed above boundedness of the electrostatic self-energy  $\mathcal{E}_{es}$  appears to be completely equivalent to the existence of so-called intrinsic Poincaré type "tensions", analyzed in [30, 111, 60], and to the existence of a special compensating Coulomb "pressure", suggested in [144], guaranteeing the observable electron stability.



**2.3. Comments.** The charged particle radiation problem, revisited in this Section, allows to conceive the following explanation of the charged particle mass as that of a compact and stable object which should be exerted by a vacuum field interaction energy potential  $\bar{W} : M^4 \rightarrow \mathbb{R}$  of negative sign as follows from (2.19). The latter can be satisfied iff the expression (2.18) holds, thereby imposing on the intrinsic charged particle structure [106] some nontrivial geometrical constraints. Moreover, as follows from the physically observed particle mass expressions (2.19) the electrostatic potential energy, being of the repulsive force origin, does contribute into the full mass as its main energy component.

There exist different relativistic generalizations of the force expression (2.18), which suffer the same common physical inconsistency related with the no radiation effect of a charged particle at uniform motion.

Another deeply related problem to the radiation reaction force analyzed above is the search for explanation to the Wheeler and Feynman reaction radiation mechanism, called the absorption radiation theory, strongly based on the Mach type interaction of a charged particle with the ambient vacuum electromagnetic medium. Concerning this problem, one can also observe some its relationships with the one devised here within the vacuum field theory approach, but this question needs a more detailed and extended analysis.

### 3. A CHARGED POINT PARTICLE DYNAMICS AND A HADRONIC STRING MODEL ANALYSIS

**3.1. The classical least action principle in electrodynamics.** The classical least action principle is considered in modern physics as a fundamental tool for deriving true and physically sound equations governing the dynamics of the corresponding physical objects. By means of this principle there has been described many physical models [97, 56, 57, 82] including those of classical mechanics, electrodynamics and Einsteinian gravity theory. As it was mentioned in many classical manuals [82, 97, 39, 28, 29], a suitable and physically motivated method of choosing the corresponding Lagrangian functions appears nowadays to be open for studying. Application of the least action principle is strongly complicated by inconsistencies often accompanying the derived physical statements which are well understood and checked by means of other physical theories. In particular, in modern electrodynamics of a charged point particle moving under influence of an external electromagnetic field, there is well known misreading [97, 57, 82] related with the charged point particle energy expression. Namely, the latter being obtained by means of the classical least action principle, gives rise to the charged particle "dynamical" mass expression not depending on the external potential energy. This fact was also discussed in the literature, for instance in [33], where there are also described other physically reasonable examples. Taking this into account and being motivated by R. Feynman's considerations of the problem in [57, 49, 50, 99] and a recently devised vacuum field theory approach [24, 126, 20] to physically reasonable formulating the corresponding least action principle for describing the charged point particle electrodynamics, we have revisited in [121] the Feynman's approach to describing the electrodynamics of a charged point particle and stated its complete legacy. Based on this experience we apply, within the present work, a slightly corrected vacuum field theory approach to describing the electrodynamics both of a charged point particle and a charged one-dimensional string under influence of an external electromagnetic field, generated by a moving charged point particle with respect to some laboratory reference frame.

It is well known [10, 57, 97, 116] that the relativistic least action principle for a charged point particle  $\xi$  in the Minkowski space-time  $M^4 \simeq \mathbb{R} \times \mathbb{E}^3$  can be formulated on a time interval  $[t_1, t_2] \subset \mathbb{R}$  (in the light speed units) as

$$\begin{aligned}
 \delta S^{(t)} &= 0, \quad S^{(t)} := \int_{\tau(t_1)}^{\tau(t_2)} (-m_0 d\tau - \langle \xi \mathcal{A}, dx \rangle_{M^4}) = \\
 (3.1) \quad &= \int_{s(t_1)}^{s(t_2)} (-m_0 \langle \dot{x}, \dot{x} \rangle_{M^4}^{1/2} - \langle \xi \mathcal{A}, \dot{x} \rangle_{M^4}) ds.
 \end{aligned}$$

Here  $\delta x(s(t_1)) = 0 = \delta x(s(t_2))$  are the boundary constraints,  $m_0 \in \mathbb{R}_+$  is the so-called particle rest mass, the 4-vector  $x := (t, r) \in M^4$  is the particle location in  $M^4$ ,  $\dot{x} := dx/ds \in T(M^4)$  is the particle 4-vector velocity with respect to a laboratory reference frame  $\mathcal{K}$ , parameterized by means of the Minkowski space-time parameters  $(s(t), r) \in M^4$  and related to each other by means of the

infinitesimal Lorentz interval relationship

$$(3.2) \quad d\tau := \langle dx, dx \rangle_{M^4}^{1/2} := ds \langle \dot{x}, \dot{x} \rangle_{M^4}^{1/2},$$

$\mathcal{A} \in T^*(M^4)$  is an external electromagnetic 4-vector potential, satisfying the classical Maxwell equations [116, 97, 57], the sign  $\langle \cdot, \cdot \rangle_{\mathcal{H}}$  means, in general, the corresponding scalar product in a finite-dimensional vector space  $\mathcal{H}$  and  $T(M^4), T^*(M^4)$  are, respectively, the tangent and cotangent spaces [1, 7, 148, 20] to the Minkowski space  $M^4$ . In particular,  $\langle x, x \rangle_{M^4} := t^2 - \langle r, r \rangle_{\mathbb{E}^3}$  for any  $x := (t, r) \in M^4$ .

The subintegral expression in (3.1)

$$(3.3) \quad \mathcal{L}^{(t)} := -m_0 \langle \dot{x}, \dot{x} \rangle_{M^4}^{1/2} - \langle \xi \mathcal{A}, \dot{x} \rangle_{M^4}$$

is the related Lagrangian function, whose first part is proportional to the particle world line length with respect to the proper rest reference frame  $\mathcal{K}_r$  and the second part is proportional to the pure electromagnetic particle-field interaction with respect to the Minkowski laboratory reference frame  $\mathcal{K}$ . Moreover, the positive rest mass parameter  $m_0 \in \mathbb{R}_+$  is introduced into (3.3) as an external physical ingredient, also describing the point particle with respect to the proper rest reference frame  $\mathcal{K}_r$ . The electromagnetic 4-vector potential  $\mathcal{A} \in T^*(M^4)$  is at the same time expressed as a 4-vector, constructed and measured with respect to the Minkowski laboratory reference frame  $\mathcal{K}$  that looks, from physical point of view, enough controversial, since the action functional (3.1) is forced to be extremal [33, 71, 78, 114, 108, 107, 149, 154] with respect to the laboratory reference frame  $\mathcal{K}$ . This, in particular, means that the real physical motion of our charged point particle, being realized with respect to the proper rest reference frame  $\mathcal{K}_r$ , somehow depends on an external observation data with respect to the occasionally chosen laboratory reference frame  $\mathcal{K}$ . This aspect was never discussed in the physical literature except for very interesting reasonings by R. Feynman in [57], who argued that the relativistic expression for the classical Lorentz force has a physical sense only with respect to the Euclidean rest reference frame  $\mathcal{K}_r$  variables  $(\tau, r) \in \mathbb{E}^4$  related to the Minkowski laboratory reference frame  $\mathcal{K}$  parameters  $(r, t) \in M^4$  by means of the infinitesimal relationship

$$(3.4) \quad d\tau := \langle dx, dx \rangle_{M^4}^{1/2} = dt(1 - |u|^2)^{1/2},$$

where  $u := dr/dt \in T(\mathbb{R}^3)$  is the point particle velocity with respect to the reference frame  $\mathcal{K}$ .

It is worth to point out here that to be correct, it would be necessary to include into the action functional the additional part describing the electromagnetic field itself. But this part is not taken into account, since there is generally assumed [88, 87, 97, 57, 36, 155, 34] that the charged particle influence on the electromagnetic field is negligible. This is true, if the particle charge value  $\xi$  is very small but the support  $\text{supp} \mathcal{A} \subset M^4$  of the electromagnetic 4-vector potential is compact. It is clear that in the case of two interacting with each other charged particles the above assumption cannot be applied, as it is necessary to take into account the relative motion of two particles and the respectively changing interaction energy. This aspect of the action functional choice problem appears to be very important when one tries to analyze the related Lorentz type forces exerted by charged particles on themselves. We will return to this problem in a separate section below.

Having calculated the least action condition (3.1), we easily obtain from (3.3) the classical relativistic dynamical equations

$$(3.5) \quad \begin{aligned} dP/ds &: = -\partial \mathcal{L}^{(t)} / \partial x = -\nabla_x \langle \xi \mathcal{A}, \dot{x} \rangle_{M^4}, \\ P &: = -\partial \mathcal{L}^{(t)} / \partial \dot{x} = m_0 \dot{x} \langle \dot{x}, \dot{x} \rangle_{M^4}^{-1/2} + \xi \mathcal{A}, \end{aligned}$$

where we denoted by  $\nabla_x$  the gradient operator with respect to the variable  $x \in M^4$  and by  $P \in T^*(M^4)$  the common particle-field momentum of the interacting system.

Now at  $s = t \in \mathbb{R}$  by means of the standard infinitesimal change of variables (3.4), we can easily obtain from (3.5) the classical Lorentz force expression

$$(3.6) \quad dp/dt = \xi E + \xi u \times B$$

with the relativistic particle momentum and mass

$$(3.7) \quad p := mu, \quad m := m_0(1 - |u|^2)^{-1/2}, \quad |u|^2 := \langle u, u \rangle_{\mathbb{E}^3},$$

respectively, the electric field

$$(3.8) \quad E := -\partial A / \partial t - \nabla \varphi$$

and the magnetic field

$$(3.9) \quad B := \nabla \times A,$$

where we have expressed the electromagnetic 4-vector potential as  $\mathcal{A} = (\varphi, A) \in T^*(M^4)$ .

The Lorentz force (3.6), owing to our preceding assumption, means the force exerted by the external electromagnetic field on our charged point particle, whose charge  $\xi$  is so negligible that it does not exert the influence on the field. This fact becomes very important if we try to make use of the Lorentz force expression (3.6) for the case of two charged interacting with each other particles, since then one cannot assume that our charge  $\xi$  exerts negligible influence on other charged particle. Thus, the corresponding Lorentz force between two charged particles should be suitably altered. Nonetheless, the modern physics [28, 29, 40, 97, 32, 38, 80, 11, 139, 8] did not make this naturally needed Lorentz force modification and there is used the classical expression (3.6). This situation was observed and analyzed concerning the related physical aspects in [136], and there was shown that the electromagnetic Lorentz force between two moving charged particles can be modified in such a way that it ceases to be dependent on their relative motion contrary to the classical relativistic case.

To our regret, the least action principle approach to analyze the Lorentz force structure was in [136] completely ignored and that gave rise to some incorrect and physically not motivated statements concerning mathematical physics backgrounds of the modern electrodynamics. To make the problem more transparent we will analyze it in the section below from the vacuum field theory approach recently devised in [19, 24, 125, 126].

**3.2. The classical least action principle revisited.** Consider the least action principle (3.1) and observe that the extremality condition

$$(3.10) \quad \delta S^{(t)} = 0, \quad \delta x(s(t_1)) = 0 = \delta x(s(t_2)),$$

is calculated with respect to the laboratory reference frame  $\mathcal{K}$ , whose point particle coordinates  $(t, r) \in M^4$  are parameterized by means of an arbitrary parameter  $s \in \mathbb{R}$  owing to expression (3.2). Recalling now the definition of the invariant proper rest reference frame  $\mathcal{K}_r$  time parameter (3.4), we obtain that at the critical parameter value  $s = \tau \in \mathbb{R}$  the action functional (3.1) on the fixed interval  $[\tau_1, \tau_2] \subset \mathbb{R}$  turns into

$$(3.11) \quad S^{(t)} = \int_{\tau_1}^{\tau_2} (-m_0 - \langle \xi \mathcal{A}, \dot{x} \rangle_{M^4}) d\tau$$

under the *additional constraint*

$$(3.12) \quad \langle \dot{x}, \dot{x} \rangle_{M^4}^{1/2} = 1,$$

where, by definition,  $\dot{x} := dx/d\tau$ ,  $\tau \in \mathbb{R}$ .

The expressions (3.11) and (3.12) need some comments since the corresponding (3.11) Lagrangian function

$$(3.13) \quad \mathcal{L}^{(t)} := -m_0 - \langle \xi \mathcal{A}, \dot{x} \rangle_{M^4}$$

depends virtually only on the unobservable rest mass parameter  $m_0 \in \mathbb{R}_+$  and, evidently, it has no direct impact on the resulting particle dynamical equations following from the condition  $\delta S^{(t)} = 0$ . Nonetheless, the rest mass springs up as a suitable Lagrangian multiplier owing to the imposed constraint (3.12). To demonstrate this, consider the extended Lagrangian function (3.13) in the form

$$(3.14) \quad \mathcal{L}_\lambda^{(t)} := -m_0 - \langle \xi \mathcal{A}, \dot{x} \rangle_{M^4} - \lambda (\langle \dot{x}, \dot{x} \rangle_{M^4}^{1/2} - 1),$$

where  $\lambda \in \mathbb{R}$  is a suitable Lagrangian multiplier. The resulting Euler equations look as follows

$$(3.15) \quad \begin{aligned} P_r &:= \partial \mathcal{L}_\lambda^{(t)} / \partial \dot{r} = \xi A + \lambda \dot{r}, \quad P_t := \partial \mathcal{L}_\lambda^{(t)} / \partial \dot{t} = -\xi \varphi - \lambda \dot{t}, \\ \partial \mathcal{L}_\lambda^{(t)} / \partial \lambda &= \langle \dot{x}, \dot{x} \rangle_{M^4}^{1/2} - 1 = 0, \quad dP_r / d\tau = \xi \nabla \langle A, \dot{r} \rangle_{\mathbb{E}^3} - \xi \dot{t} \nabla \varphi, \\ dP_t / d\tau &= \xi \langle \partial A / \partial t, \dot{r} \rangle_{\mathbb{E}^3} - \xi \dot{t} \partial \varphi / \partial t, \end{aligned}$$

giving rise, owing to relationship (3.4), to the following dynamical equations:

$$(3.16) \quad \frac{d}{dt}(\lambda u \dot{t}) = \xi E + \xi u \times B, \quad \frac{d}{dt}(\lambda \dot{t}) = \xi \langle E, u \rangle_{\mathbb{E}^3},$$

where we denoted by

$$(3.17) \quad E := -\partial A / \partial t - \nabla \varphi, \quad B = \nabla \times A$$

the corresponding electric and magnetic fields. As a simple consequence of (3.16) one obtains

$$(3.18) \quad \frac{d}{dt} \ln(\lambda \dot{t}) + \frac{d}{dt} \ln(1 - |u|^2)^{1/2} = 0,$$

being equivalent for all  $t \in \mathbb{R}$ , owing to relationship (3.4), to the relationship

$$(3.19) \quad \lambda \dot{t} (1 - |u|^2)^{1/2} = \lambda := m_0,$$

where  $m_0 \in \mathbb{R}_+$  is a constant, which could be interpreted as the rest mass of our charged point particle  $\xi$ . Really, the first equation of (3.16) can be rewritten as

$$(3.20) \quad dp/dt = \xi E + \xi u \times B,$$

where we denoted

$$(3.21) \quad p := mu, \quad m := \lambda \dot{t} = m_0 (1 - |u|^2)^{-1/2},$$

coinciding exactly with that of (3.4).

Thereby, we retrieved here all the results obtained in the section above, making use of the action functional (3.11), expressed with respect to the rest reference frame  $\mathcal{K}_r$  under constraint (3.12). During these derivations, we were faced with a very delicate inconsistency property of definition of the action functional  $S^{(t)}$ , defined with respect to the rest reference frame  $\mathcal{K}_r$ , but depending on the external electromagnetic potential function  $\mathcal{A} : M^4 \rightarrow T^*(M^4)$ , constructed exceptionally with respect to the laboratory reference frame  $\mathcal{K}$ . Namely, this potential function, as a physical observable quantity, is defined and, respectively, measurable only with respect to the fixed laboratory reference frame  $\mathcal{K}$ . This, in particular, means that a physically reasonable action functional should be constructed by means of an expression strongly calculated within the laboratory reference frame  $\mathcal{K}$  by means of coordinates  $(t, r) \in M^4$  and later suitably transformed subject to the rest reference frame  $\mathcal{K}_r$  coordinates  $(\tau, r) \in \mathbb{E}^4$ , respective for the real charged point particle  $\xi$  motion. Thus, the corresponding dual action functional, in reality, should be from the very beginning written as

$$(3.22) \quad S^{(\tau)} = \int_{t(\tau_1)}^{t(\tau_2)} (-\langle \xi \mathcal{A}, \dot{x} \rangle_{M^4}) dt,$$

where  $\dot{x} := dx/dt$ ,  $t \in \mathbb{R}$ , being calculated on some time interval  $[t(\tau_1), t(\tau_2)] \subset \mathbb{R}$ , suitably related with the proper motion of the charged point particle  $\xi$  on the true time interval  $[\tau_1, \tau_2] \subset \mathbb{R}$  with respect to the rest reference frame  $\mathcal{K}_r$  and whose charge value is assumed so negligible that it exerts no influence on the external electromagnetic field. Now the problem arises: how to compute correctly the variation  $\delta S^{(\tau)} = 0$  of the action functional (3.22)?

To reply to this question we will turn to the R. Feynman reasonings from [57], where he argued, when deriving the relativistic Lorentz force expression, that the real charged particle dynamics can be determined physically not ambiguously only with respect to the rest reference frame time parameter. Namely, R. Feynman wrote: "...we calculate a growth  $\Delta x$  for a small time interval  $\Delta t$ . But in the other reference frame the interval  $\Delta t$  may correspond to changing both  $t'$  and  $x'$ , thereby at the change of the only  $t'$  the suitable change of  $x$  will be other... Making use of the quantity  $d\tau$  one can determine a good differential operator  $d/d\tau$ , as it is invariant with respect to the Lorentz reference frames of transformations". This means that if our charged particle  $\xi$  moves in the Minkowski space  $M^4$  during the time interval  $[t_1, t_2] \subset \mathbb{R}$  with respect to the laboratory reference frame  $\mathcal{K}$ , its proper real and invariant time of motion with respect to the rest reference frame  $\mathcal{K}_r$  will be respectively  $[\tau_1, \tau_2] \subset \mathbb{R}$ .

As a corollary of the R. Feynman reasonings, we arrive at the necessity to rewrite the action functional (3.22) as

$$(3.23) \quad S^{(\tau)} = \int_{\tau_1}^{\tau_2} (-\langle \xi A, \dot{x} \rangle_{M^4}) d\tau, \quad \delta x(\tau_1) = 0 = \delta x(\tau_2),$$

where  $\dot{x} := dx/d\tau$ ,  $\tau \in \mathbb{R}$ , under the *additional constraint*

$$(3.24) \quad \langle \dot{x}, \dot{x} \rangle_{M^4}^{1/2} = 1,$$

being equivalent to the infinitesimal transformation (3.4). Since the functional (3.23) is still computed in the laboratory reference frame  $\mathcal{K}$  variables  $x \in M^4$ , it is necessary to represent it with respect to the rest reference frame  $\mathcal{K}_r$ . Simultaneously the proper time interval  $[\tau_1, \tau_2] \subset \mathbb{R}$  is mapped on the time interval  $[t_1, t_2] \subset \mathbb{R}$  by means of the infinitesimal transformation

$$(3.25) \quad dt = d\tau(1 + |\dot{r}|^2)^{1/2},$$

where  $\dot{r} := dr/d\tau$ ,  $\tau \in \mathbb{R}$ . Thus, we can now pose the *true least action problem* equivalent to (3.23) and (3.24) as

$$(3.26) \quad \delta S^{(\tau)} = 0, \quad \delta r(\tau_1) = 0 = \delta r(\tau_2),$$

where the functional

$$(3.27) \quad S^{(\tau)} = \int_{\tau_1}^{\tau_2} [-\bar{W}(1 + |\dot{r}|^2)^{1/2} + \langle \xi A, \dot{r} \rangle_{\mathbb{E}^3}] d\tau$$

is characterized by the Lagrangian function

$$(3.28) \quad \mathcal{L}^{(\tau)} := -\bar{W}(1 + |\dot{r}|^2)^{1/2} + \langle \xi A, \dot{r} \rangle_{\mathbb{E}^3}.$$

Here we denoted, for further convenience,  $\bar{W} := \xi\varphi$ , being the suitable vacuum field [125, 126, 24, 136] potential function. The resulting Euler equation gives rise to the following relationships

$$(3.29) \quad \begin{aligned} P &: = \partial \mathcal{L}^{(\tau)} / \partial \dot{r} = -\bar{W} \dot{r} (1 + |\dot{r}|^2)^{-1/2} + \xi A, \\ dP/d\tau &: = \partial \mathcal{L}^{(\tau)} / \partial r = -\nabla \bar{W} (1 + |\dot{r}|^2)^{1/2} + \xi \nabla \langle A, \dot{r} \rangle_{\mathbb{E}^3}. \end{aligned}$$

Now making use once more of the infinitesimal transformation (3.25) and the crucial *dynamical particle mass* definition [125, 24, 136, 149] (in the light speed units )

$$(3.30) \quad m := -\bar{W},$$

we can easily rewrite equations (3.29) with respect to the parameter  $t \in \mathbb{R}$  as the classical relativistic Lorentz force:

$$(3.31) \quad dp/dt = \xi E + \xi u \times B,$$

where we denoted

$$(3.32) \quad \begin{aligned} p &: = mu, & u &:= dr/dt, \\ B &: = \nabla \times A, & E &:= -\xi^{-1} \nabla \bar{W} - \partial A / \partial t. \end{aligned}$$

Thus, we obtained once more the relativistic Lorentz force expression (3.31), but slightly different from (3.6), since the classical relativistic momentum of (3.7) does not completely coincide with our modified relativistic momentum expression

$$(3.33) \quad p = -\bar{W}u,$$

depending strongly on the scalar vacuum field potential function  $\bar{W} : M^4 \rightarrow \mathbb{R}$ . But if we recall here that our action functional (3.23) was written under the assumption that the particle charge value  $\xi$  is negligible and not exerting the essential influence on the electromagnetic field source, we can make use of the result before obtained in [19, 24, 125, 126, 136], that the vacuum field potential function  $\bar{W} : M^4 \rightarrow \mathbb{R}$ , owing to (3.31)-(3.33), satisfies as  $\xi \rightarrow 0$  the dynamical equation

$$(3.34) \quad d(-\bar{W}u)/dt = -\nabla \bar{W},$$

whose solution at the stationary condition  $\partial \bar{W} / \partial t = 0$  will be exactly the expression

$$(3.35) \quad -\bar{W} = m_0(1 - |u|^2)^{-1/2}, \quad m_0 = -\bar{W}|_{u=0}.$$

Thereby, we have arrived, owing to (3.35) and (3.33), to the almost full *coincidence* of our result (3.31) for the relativistic Lorentz force with that of (3.6) under the condition  $\xi \rightarrow 0$ .

The results obtained above and related inferences we will formulate as the following proposition.

**Proposition 3.1.** *Under the assumption of the negligible influence of a charged point particle  $\xi$  on an external electromagnetic field source a true physically reasonable action functional can be represented by expression (3.22), being equivalently defined with respect to the rest reference frame  $\mathcal{K}_r$  in the form (3.23), (3.24). The resulting relativistic Lorentz force (3.31) coincides almost exactly with that of (3.6), obtained from the classical Einstein type action functional (3.1), but the momentum expression (3.33) differs from the classical expression (3.7), taking into account the related vacuum field potential interaction energy impact.*

As an important corollary we make the following.

*Corollary 3.2.* The Lorentz force expression (3.31) should be, in due course, corrected in the case when the weak charge  $e$  influence assumption made above does not hold.

*Remark 3.3.* Concerning the infinitesimal relationship (3.25) one can observe that it reflects the Euclidean nature of transformations  $\mathbb{R} \ni t \rightleftharpoons \tau \in \mathbb{R}$ .

In spite of the results obtained above by means of two different least action principles (3.1) and (3.23), we must claim here that the first one possesses some logical controversies, which may give rise to unpredictable, unexplainable and even nonphysical effects. Amongst these controversies we mention:

- i)* the definition of Lagrangian function (3.3) as an expression, depending on the external and undefined rest mass parameter with respect to the rest reference frame  $\mathcal{K}_r$  time  $\tau \in \mathbb{R}$ , but serving as a variational integrand with respect to the laboratory reference frame  $\mathcal{K}$  time  $t \in \mathbb{R}$ ;
- ii)* the least action condition (3.1) is calculated with respect to the fixed boundary conditions at the ends of a time interval  $[t_1, t_2] \subset \mathbb{R}$ , thereby the resulting dynamics becomes strongly dependent on the chosen laboratory reference frame  $\mathcal{K}$ , which is, following the Feynman arguments [57], physically unreasonable;
- iii)* the resulting relativistic particle mass and its energy depend only on the particle velocity in the laboratory reference frame  $\mathcal{K}$ , not taking into account the present vacuum field potential energy of a charged point particle, exerting not trivial action on the particle motion;
- iv)* the assumption concerning the negligible influence of a charged point particle on the external electromagnetic field source is also physically inconsistent.

#### 4. A CHARGED POINT PARTICLE ELECTRODYNAMICS WITHIN THE VACUUM FIELD THEORY

##### APPROACH

**4.1. A free charged point particle in the vacuum medium.** We start now from the following action functional for a charged point particle  $\xi$  moving with velocity  $u := dr/dt \in \mathbb{E}^3$  under an external stationary vacuum field potential  $\bar{W} : M^4 \rightarrow \mathbb{R}$  with respect to a laboratory reference frame  $\mathcal{K}$ :

$$(4.1) \quad S^{(\tau)} := - \int_{t(\tau_2)}^{t(\tau_1)} \bar{W} dt,$$

being defined on the time interval  $[t(\tau_1), t(\tau_2)] \subset \mathbb{R}$ . The vacuum field potential function  $\bar{W} : M^4 \rightarrow \mathbb{R}$  satisfies the condition  $\partial \bar{W} / \partial t = 0$  and characterizes the intrinsic properties of the vacuum medium and its interaction with a charged point particle  $\xi$ , jointly with the Euclidean rest reference frame  $\mathcal{K}_r$  relationship

$$(4.2) \quad dt = \langle \dot{\eta}, \dot{\eta} \rangle_{\mathbb{E}^4}^{1/2} d\tau,$$

where  $\eta := (\tau, r) \in \mathbb{E}^4$  is a charged point particle position 4-vector with respect to the proper rest reference frame  $\mathcal{K}_r$ ,  $\dot{\eta} := d\eta/d\tau$ ,  $\tau \in \mathbb{R}$ . As the real dynamics of our charged point particle

$e$  depends strongly only on the time interval  $[\tau_1, \tau_2] \subset \mathbb{R}$  of its own motion subject to the rest reference frame  $\mathcal{K}_r$ , we need to calculate the extremality condition

$$(4.3) \quad \delta S^{(\tau)} = 0, \quad \delta r(\tau_1) = 0 = \delta r(\tau_2).$$

As action functional (4.1) is equivalent, owing to (4.2) or (3.25), to the following:

$$(4.4) \quad S^{(\tau)} := - \int_{\tau_2}^{\tau_1} \bar{W}(1 + |\dot{r}|^2)^{1/2} d\tau,$$

where, by definition,  $\dot{r} := dr/d\tau$ ,  $|\dot{r}|^2 := \langle \dot{r}, \dot{r} \rangle_{\mathbb{E}^3}$ ,  $\tau \in \mathbb{R}$ , from (4.4) and (4.3) one easily obtains that

$$(4.5) \quad p := -\bar{W}\dot{r}(1 + |\dot{r}|^2)^{-1/2}, \quad dp/d\tau = -\nabla \bar{W}(1 + |\dot{r}|^2)^{1/2}.$$

Taking into account once more relationship (3.25) we can rewrite (4.5) equivalently as

$$(4.6) \quad dp/dt = -\nabla \bar{W}, \quad p := -\bar{W}u.$$

If to take into account the dynamic mass definition (3.30), equation (4.6) turns into the Newton dynamical expression

$$(4.7) \quad dp/dt = -\nabla \bar{W}, \quad p = mu.$$

Having observed now that equation (4.7) is completely equivalent to equation (3.34), we obtain right away from (3.35) that the particle mass

$$(4.8) \quad m = m_0(1 - |u|^2)^{-1/2},$$

where

$$(4.9) \quad m_0 := -\bar{W}|_{u=0}$$

is the so-called particle rest mass. Moreover, since the corresponding (4.4) Lagrangian function

$$(4.10) \quad \mathcal{L}^{(\tau)} := -\bar{W}(1 + |\dot{r}|^2)^{1/2}$$

is not degenerate, we can easily construct [1, 7, 20, 24] the related conservative Hamiltonian function

$$(4.11) \quad \mathcal{H}^{(\tau)} = -(\bar{W}^2 - |p|^2)^{1/2},$$

where  $|p|^2 := \langle p, p \rangle_{\mathbb{E}^3}$ , satisfying the canonical Hamiltonian equations

$$(4.12) \quad dr/d\tau = \partial \mathcal{H}^{(\tau)} / \partial p, \quad dp/d\tau = -\partial \mathcal{H}^{(\tau)} / \partial r$$

and conservation conditions

$$(4.13) \quad d\mathcal{H}^{(\tau)} / dt = 0 = d\mathcal{H}^{(\tau)} / d\tau$$

for all  $\tau, t \in \mathbb{R}$ . Thereby, the quantity

$$(4.14) \quad \mathcal{E} := (\bar{W}^2 - |p|^2)^{1/2}$$

can be naturally interpreted as the point particle total energy.

It is important to note here that energy expression (4.14) takes into account both kinetic and potential energies, but the particle dynamic mass (4.8) depends only on its velocity, reflecting its free motion in vacuum. Moreover, since the vacuum potential function  $\bar{W} : M^4 \rightarrow \mathbb{R}$  is not, in general, constant, we claim that the motion of our particle  $e$  with respect to the laboratory reference frame  $\mathcal{K}$  is not, in general, linear and is with not constant velocity, - the situation, which was already discussed before by R. Feynman in [57]. Thus, we obtained the classical relativistic mass dependence on the freely moving particle velocity (4.8), taking into account both the nonconstant vacuum potential function  $\bar{W} : M^4 \rightarrow \mathbb{R}$  and the particle velocity  $u \in \mathbb{E}^3$ .

We would also like to mention here that the vacuum potential function  $\bar{W} : M^4 \rightarrow \mathbb{R}$  itself should be simultaneously found by means of a suitable solution to the Maxwell equation  $\partial^2 W / \partial t^2 - \Delta W = \rho$ , where  $\rho \in \mathbb{R}$  is an ambient charge density and, by definition,  $\bar{W}(t, r(t)) := \lim_{r \rightarrow r(t)} W(r, t)$ , with  $r(t) \in \mathbb{E}^3$ , being the position of the charged point particle at a time moment  $t \in \mathbb{R}$ . A more detailed description [125] of the vacuum field potential  $W : M^4 \rightarrow \mathbb{R}$ , developing the ideas of [149] and characterizing the vacuum medium structure, is given in the Supplement.

We return now to expression (4.1) and rewrite it in the following invariant form

$$(4.15) \quad S^{(\tau)} = - \int_{s(\tau_1)}^{s(\tau_2)} \bar{W} \langle \dot{\eta}, \dot{\eta} \rangle_{\mathbb{E}^4}^{1/2} ds,$$

where, by definition,  $s \in \mathbb{R}$  parameterizes the particle world line related with the laboratory reference frame  $\mathcal{K}$  time parameter  $t \in \mathbb{R}$  by means of the Euclidean infinitesimal *relationship*

$$(4.16) \quad dt := \langle \dot{\eta}, \dot{\eta} \rangle_{\mathbb{E}^4}^{1/2} ds.$$

It is easy to observe that at  $s = t \in \mathbb{R}$  the functional (4.15) turns into (3.11) and the *constraint* (3.12), at  $s = \tau \in \mathbb{R}$  it transforms into (4.1) and the *relationship* (4.2). The action functional (4.15) is to be supplemented naturally with the boundary conditions

$$(4.17) \quad \delta\eta(s(\tau_1)) = 0 = \delta\eta(s(\tau_2)),$$

which are, obviously, completely equivalent to those of (4.3), since the mapping  $\mathbb{R} \ni s \rightleftharpoons t \in \mathbb{R}$ , owing to definition (4.16) is one-to-one.

Having calculated the least action condition  $\delta S^{(\tau)} = 0$  under constraints (4.17), one easily obtains the same equation (4.6) and relationships (4.8), (4.14) for the particle dynamical mass and its conservative energy, respectively.

#### 4.2. A charged point particle electrodynamics under external electromagnetic field.

We would like to generalize the results obtained above for a free point particle in the vacuum medium for the case of a charged point particle  $\xi$  interacting with an external charged point particle  $\xi_f$ , moving with respect to a laboratory reference frame  $\mathcal{K}$  with the velocity  $u_f \in \mathbb{E}^3$ . Within the vacuum field theory approach, devised in [125, 126, 24], it is natural to reduce the formulated problem to that considered above, having introduced the reference frame  $\mathcal{K}'_f$  moving with respect to the reference frame  $\mathcal{K}'_f$  with the same velocity as that of the external charged point particle  $\xi_f$ . Thus, if the external charged particle  $\xi_f$ , considered with respect to the laboratory reference  $\mathcal{K}'_f$ , will be in rest, the test charged point particle  $\xi$  will be moving with the resulting velocity  $u' - u'_f \in T(\mathbb{R}^3)$ , where, by definition,  $u' := dr/dt'$ ,  $u'_f := dr_f/dt'$  are the corresponding velocities of these charged point particles  $\xi$  and  $\xi_f$  with respect to the moving reference frame  $\mathcal{K}'_f$ , parametrized by the Euclidean variables  $(t', r_f) \in \mathbb{E}^4$ , and which are under the influence of the corresponding external vacuum field potential  $\bar{W}' \in \mathbb{R}$  related via the Lorentz transform

$$(4.18) \quad \bar{W}'(1 - |u_f|^2)^{1/2} = \bar{W}$$

to that with respect to the laboratory reference frame  $\mathcal{K}$ . As a result of these reasonings we can write down the following action functional expression

$$(4.19) \quad S^{(\tau)} = - \int_{t'(\tau_1)}^{t'(\tau_2)} \bar{W}' dt' = \int_{s(\tau_1)}^{s(\tau_2)} \bar{W}' \langle \eta'_f, \eta'_f \rangle_{\mathbb{E}^4}^{1/2} ds,$$

where, by definition,  $\dot{\eta}_f := (\dot{\tau}, \dot{r} - \dot{r}_f) \in T(\mathbb{E}^4)$  is the charged point particle  $\xi$  four-vector velocity with respect to the rest reference frame  $\mathcal{K}_r$  and calculated subject to the corresponding particle world line parameter  $s \in \mathbb{R}$ , being infinitesimally related to the moving reference frame  $\mathcal{K}'_f$  time parameter  $t' \in \mathbb{R}$  as

$$(4.20) \quad dt' := \langle \dot{\eta}_f, \dot{\eta}_f \rangle_{\mathbb{E}^4}^{1/2} ds.$$

The boundary conditions for functional (4.19) are taken naturally in the form

$$(4.21) \quad \delta\eta(s(\tau_1)) = 0 = \delta\eta(s(\tau_2)),$$

where  $\eta = (\tau, r) \in \mathbb{E}^4$ . The least action condition  $\delta S^{(\tau)} = 0$  jointly with (4.21) gives rise to the following equations:

$$(4.22) \quad \begin{aligned} P & : = \partial \mathcal{L}^{(\tau)} / \partial \dot{\eta} = -\bar{W}' \dot{\eta}_f < \dot{\eta}_f, \dot{\eta}_f \rangle_{\mathbb{E}^4}^{-1/2}, \\ dP/ds & : = \partial \mathcal{L}^{(\tau)} / \partial \eta = -\nabla_\eta \bar{W}' < \dot{\eta}_f, \dot{\eta}_f \rangle_{\mathbb{E}^4}^{1/2}, \end{aligned}$$

where the Lagrangian function equals

$$(4.23) \quad \mathcal{L}^{(\tau)} := -\bar{W}' < \dot{\eta}_f, \dot{\eta}_f \rangle_{\mathbb{E}^4}^{1/2}.$$

Having now defined, owing to the relationship (4.18), the charged point particle  $\xi$  momentum  $p \in T^*(\mathbb{E}^3)$  as

$$(4.24) \quad p := -\bar{W}' \dot{r} < \dot{\eta}_f, \dot{\eta}_f \rangle_{\mathbb{E}^4}^{-1/2} = -\bar{W}' u' = -\bar{W} u$$



and the induced external magnetic vector potential  $A \in T^*(\mathbb{E}^3)$  as

$$(4.25) \quad \xi A := \bar{W}' \dot{r}_f < \dot{\eta}_f, \dot{\eta}_f >_{\mathbb{E}^4}^{-1/2} = \bar{W}' u'_f = \bar{W} u_f,$$

we obtain, owing to relationship (4.20), the equality

$$(4.26) \quad d(p + \xi A)/dt = -\nabla W(1 - |u_f|_{\mathbb{E}^3}^2),$$

or the final relativistic Lorentz type force expression

$$(4.27) \quad dp/dt = \xi E + \xi u \times B - \xi \nabla < u - u_f, A >_{\mathbb{E}^3},$$

where we denoted, by definition,

$$(4.28) \quad E := -\xi^{-1} \nabla \bar{W} - \partial A / \partial t, \quad B = \nabla \times A,$$

being, respectively, the external electric and magnetic fields, acting on the charged point particle  $\xi$ .

The result (4.27) contains the additional Lorentz force component

$$(4.29) \quad F_c := -\xi \nabla < u - u_f, A >_{\mathbb{E}^3},$$

not present in the classical relativistic Lorentz force expressions (3.6) and (3.31), obtained before.

*Remark 4.1.* It is worthy to remark here that just recently a gradient like term, similar to that of (4.29), was obtained by R.T. Hammond in a very interesting work [73]. Yet the exact relationships between these two representations is still not established as that from [73] contains not completely specified functional parameters.

The considered above alternative classical relativistic electrodynamic model, based on the action functional (4.19) with respect to the rest reference frame  $\mathcal{K}_r$ , where the Lagrangian function is given by expression (4.23), is strongly supported by physical arguments. In particular, it demonstrates a well known physical phenomenon that an external charged particle  $\xi_f$ , moving with a velocity  $u_f \in E^3$  tending to the light velocity  $c$ ,  $|c| = 1$ , exerts no influence on the charged test particle  $\xi$  and equivalently expressed by the right-hand side of expression (4.26). The resulting Lorentz type force expression (4.27), being modified by the additional force component  $F_c := -\nabla < \xi A, u - u_f >_{\mathbb{E}^3}$ , is important for explanation [2, 31, 149] of the well known Aharonov-Bohm effect. Moreover, from (4.24) one obtains that the point particle  $\xi$  momentum

$$(4.30) \quad p = -\bar{W} u := mu,$$

where the particle mass

$$(4.31) \quad m := -\bar{W}$$

does not already coincide with the corresponding classical relativistic relationship of (3.7).

Consider now the least action condition for functional (4.19) at the critical parameter  $s = \tau \in \mathbb{R}$ :

$$(4.32) \quad \begin{aligned} \delta S^{(\tau)} &= 0, & \delta r(\tau_1) = 0 = \delta r(\tau_2), \\ S^{(\tau)} &: = - \int_{\tau_1}^{\tau_2} \bar{W}(1 + |\dot{r} - \dot{r}_f|_{\mathbb{E}^3}^2)^{1/2} d\tau. \end{aligned}$$

The resulting Lagrangian function

$$(4.33) \quad \mathcal{L}^{(\tau)} := -\bar{W}(1 + |\dot{r} - \dot{r}_f|_{\mathbb{E}^3}^2)^{1/2}$$

gives rise to the generalized momentum expression

$$(4.34) \quad P := \partial \mathcal{L}^{(\tau)} / \partial \dot{r} = -\bar{W}(\dot{r} - \dot{r}_f)(1 + |\dot{r} - \dot{r}_f|_{\mathbb{E}^3}^2)^{-1/2} := p + \xi A,$$

which makes it possible to construct [1, 7, 119, 20] the corresponding Hamiltonian function as

$$(4.35) \quad \begin{aligned} \mathcal{H} &: = < P, \dot{r} >_{\mathbb{E}^3} - \mathcal{L}^{(\tau)} = -(\bar{W}^2 - |p + \xi A|_{\mathbb{E}^3}^2)^{1/2} - \\ &- < p + \xi A, \xi A >_{\mathbb{E}^3} (\bar{W}^2 - |p + \xi A|_{\mathbb{E}^3}^2)^{-1/2}, \end{aligned}$$

satisfying the canonical Hamiltonian equations

$$(4.36) \quad dP/d\tau := \partial \mathcal{H} / \partial r, \quad dr/d\tau := -\partial \mathcal{H} / \partial P,$$

evolving with respect to the proper rest reference frame time  $\tau \in \mathbb{R}$  parameter. When deriving (4.35) we made use of relationship (4.20) at  $s = \tau \in \mathbb{R}$  jointly with definitions (4.24) and (4.25).

Since the Hamiltonian function (4.35) is conservative with respect to the evolution parameter  $\tau \in \mathbb{R}$ , owing to relationship (4.20) at  $s = \tau \in \mathbb{R}$  one obtains that

$$(4.37) \quad d\mathcal{H}/d\tau = 0 = d\mathcal{H}/dt$$

for all  $t, \tau \in \mathbb{R}$ . The obtained results can be formulated as the following proposition.

**Proposition 4.2.** *The charged point particle electrodynamics, related with the least action principle (4.19) and (4.21), reduces to the modified Lorentz type force equation (4.27), and is equivalent to the canonical Hamilton frame (4.36) with respect to the proper rest reference frame  $\mathcal{K}_\tau$  time parameter  $\tau \in \mathbb{R}$ . The corresponding Hamiltonian function (4.35) is a conservation law for the Lorentz type dynamics (4.27), satisfying the conditions (4.37) with respect to both reference frames  $\mathcal{K}$  and  $\mathcal{K}_\tau$  parameters  $t, \tau \in \mathbb{R}$ , respectively.*

As a corollary, the corresponding energy expression for electrodynamical model (4.27) can be defined as

$$(4.38) \quad \mathcal{E} := (\bar{W}^2 - |p + \xi A|_{\mathbb{E}^3}^2)^{1/2} + \langle p + \xi A, \xi A \rangle_{\mathbb{E}^3} (\bar{W}^2 - |p + \xi A|_{\mathbb{E}^3}^2)^{-1/2}.$$

The energy expression (4.38) obtained above is a necessary ingredient for quantizing the relativistic electrodynamics (4.27) of our charged point particle  $\xi$  under the external electromagnetic field influence.

## 5. A NEW HADRONIC STRING MODEL: THE LEAST ACTION PRINCIPLE AND RELATIVISTIC ELECTRODYNAMICS ANALYSIS WITHIN THE VACUUM FIELD THEORY APPROACH

**5.1. A new hadronic string model least action formulation.** A classical relativistic hadronic string model was first proposed in [9, 112, 64] and deeply studied in [10], making use of the least action principle and related Lagrangian and Hamiltonian formalisms. We will not discuss here this classical string model and will not comment the physical problems accompanying it, especially those related to its diverse quantization versions, but proceed to formulating a new relativistic hadronic string model, constructed by means of the vacuum field theory approach, devised in [125, 126, 24]. The corresponding least action principle is, following [10], formulated as

$$(5.1) \quad \delta S^{(\tau)} = 0, \quad S^{(\tau)} := \int_{s(\tau_1)}^{s(\tau_2)} ds \int_{\sigma_1(s)}^{\sigma_2(s)} \bar{W}(x(\eta)) (|\dot{\eta}|^2 |\eta'|^2 - \langle \dot{\eta}, \eta' \rangle_{\mathbb{E}^4}^2)^{1/2} d\sigma \wedge ds,$$

where  $\bar{W} : M^4 \rightarrow \mathbb{R}$  is a vacuum field potential function, characterizing the interaction of the vacuum medium with our charged string object, the differential 2-form  $d\Sigma^{(2)} := (|\dot{\eta}|^2 |\eta'|^2 - \langle \dot{\eta}, \eta' \rangle_{\mathbb{E}^4}^2)^{1/2} d\sigma \wedge ds = \sqrt{g(\eta)} d\sigma \wedge ds$ ,  $g(\eta) := \det(g_{ij}(\eta)|_{i,j=\overline{1,2}})$ ,  $|\dot{\eta}|^2 := \langle \dot{\eta}, \dot{\eta} \rangle_{\mathbb{E}^4}$ ,  $|\eta'|^2 := \langle \eta', \eta' \rangle_{\mathbb{E}^4}$ , being related with the induced positive definite Riemannian infinitesimal metrics  $dz^2 := \langle d\eta, d\eta \rangle_{\mathbb{E}^4} = g_{11}(\eta) d\sigma^2 + g_{12}(\eta) d\sigma ds + g_{21}(\eta) ds d\sigma + g_{22}(\eta) ds^2$  on the string, means [1, 10, 148] the infinitesimal two-dimensional world surface element, parameterized by variables  $(s, \sigma) \in \mathbb{R}^2$  and embedded into the 4-dimensional Euclidean space-time  $\mathbb{E}^4$  with coordinates  $\eta := (\tau(s, \sigma), r(s, \sigma)) \in \mathbb{E}^4$  subject to the proper rest reference frame  $\mathcal{K}_\tau$ ,  $\dot{\eta} := \partial\eta/\partial s$ ,  $\eta' := \partial\eta/\partial\sigma$  are the corresponding partial derivatives. The related boundary conditions are chosen as

$$(5.2) \quad \delta\eta(s, \sigma(s)) = 0$$

at string parameter  $\sigma(s) \in \mathbb{R}$  for all  $s \in \mathbb{R}$ . The action functional expression is strongly motivated by that constructed for the point particle action functional (4.1):

$$(5.3) \quad S^{(\tau)} := - \int_{\sigma_1}^{\sigma_2} dl(\sigma) \int_{t(\sigma, \tau_1)}^{t(\sigma, \tau_2)} \bar{W} dt(\tau, \sigma),$$

where the laboratory reference time parameter  $t(\tau, \sigma) \in \mathbb{R}$  is related to the proper rest string reference frame time parameter  $\tau \in \mathbb{R}$  by means of the standard Euclidean infinitesimal relationship

$$(5.4) \quad dt(\tau, \sigma) := (1 + |\dot{r}_\perp|^2(\tau, \sigma))^{1/2} d\tau, \quad |\dot{r}_\perp|^2 := \langle \dot{r}_\perp, \dot{r}_\perp \rangle_{\mathbb{E}^3},$$

with  $\sigma \in [\sigma_1, \sigma_2] \subset \mathbb{R}$ , being a spatial variable parameterizing the string length measure  $dl(\sigma)$  on the real axis  $\mathbb{R}$ ,  $\dot{r}_\perp(\tau, \sigma) := \hat{N} \dot{r}(\tau, \sigma) \in \mathbb{E}^3$  being the orthogonal to the string velocity component, and

$$(5.5) \quad \hat{N} := (1 - |r'|^{-2} r' \otimes r')^{-1/2}, \quad |r'|^{-2} := \langle r', r' \rangle_{\mathbb{E}^3}^{-1},$$

being the corresponding projector operator in  $\mathbb{E}^3$  on the orthogonal to the string direction, expressed for brevity by means of the standard tensor product " $\otimes$ " in the Euclidean space  $\mathbb{E}^3$ . The result of calculation of (5.3) gives rise to the following expression

$$(5.6) \quad S^{(\tau)} = - \int_{\tau_1}^{\tau_2} d\tau \int_{\sigma_1(\tau)}^{\sigma_2(\tau)} \bar{W}((|r'|^2(1 + |\dot{r}|^2) - \langle \dot{r}, r' \rangle_{\mathbb{E}^3}^2)^{1/2} d\sigma,$$

where we made use of the infinitesimal measure representation  $dl(\sigma) = \langle r', r' \rangle_{\mathbb{E}^3}^{1/2} d\sigma$ ,  $\sigma \in [\sigma_1, \sigma_2]$ . If now to introduce on the string world surface local coordinates  $(s(\tau, \sigma), \sigma) \in \mathbb{E}^2$  and the related Euclidean string position vector  $\eta := (\tau, r(s, \sigma)) \in \mathbb{E}^4$ , the string action functional reduces equivalently to that of (5.1).

Below we will proceed to Lagrangian and Hamiltonian analyzing the least action conditions for expressions (5.1) and (5.6).

**5.2. Lagrangian and Hamiltonian analysis.** First we will obtain the corresponding to (5.1) Euler equations with respect to the special [10] internal conformal variables  $(s, \sigma) \in \mathbb{E}^2$  on the world string surface, with respect to which the metrics on it becomes equal to  $dz^2 = |\eta'|^2 d\sigma^2 + |\dot{\eta}|^2 ds^2$ , where  $\langle \eta', \dot{\eta} \rangle_{\mathbb{E}^4} = 0 = |\eta'|^2 - |\dot{\eta}|^2$  are the imposed constraints, and the corresponding infinitesimal world surface measure  $d\Sigma^{(2)}$  becomes  $d\Sigma^{(2)} = |\eta'| |\dot{\eta}| d\sigma \wedge ds$ . As a result of simple calculations one finds the linear second order partial differential equation

$$(5.7) \quad \partial(\bar{W} \dot{\eta}) / \partial s + \partial(\bar{W} \eta') / \partial \sigma = \partial(|\eta'| |\dot{\eta}| \bar{W}) / \partial \sigma$$

under the suitably chosen boundary conditions

$$(5.8) \quad \eta' - \dot{\eta} \dot{\sigma} = 0$$

for all  $s \in \mathbb{R}$ . It is interesting to mention that equation (5.7) is of *elliptic type*, contrary to the case considered before in [10]. This is, evidently, owing to the fact that the resulting metrics on the string world surface is Euclidean, as we took into account that the real string motion is, in reality, realized with respect to its proper rest reference frame  $\mathcal{K}_r$ , being not dependent on the string motion observation data, measured with respect to any external laboratory reference frame  $\mathcal{K}$ . The latter can be used for physically motivated evidence of the dynamical stability of the relativistic charged string object, modeling a charged hadronic particle [8, 71, 112, 155] with non-trivial internal structure.

The differential equation (5.7) strongly depends on the vacuum field potential function  $\bar{W} : M^4 \rightarrow \mathbb{R}$ , which, as a function of the Minkowski 4-vector variable  $x := (t(s, \sigma), r) \in M^4$  of the laboratory reference frame  $\mathcal{K}$ , should be expressed as a function of the variables  $(s, \sigma) \in \mathbb{E}^2$ , making use of the infinitesimal relationship (5.4) in the following form:

$$(5.9) \quad dt = \langle \hat{N} \partial \dot{\eta} / \partial \tau, \hat{N} \partial \dot{\eta} / \partial \tau \rangle_{\mathbb{E}^3}^{1/2} \left( \frac{\partial \tau}{\partial s} ds + \frac{\partial \tau}{\partial \sigma} d\sigma \right),$$

defined on the string world surface. The function  $\bar{W} : M^4 \rightarrow \mathbb{R}$  itself should be simultaneously found, following ideas of [33, 149] and recent results of [19, 24, 125, 126], by means of a suitable solution to the Maxwell equation  $\partial^2 W / \partial t^2 - \Delta W = \rho$ , where  $\rho \in \mathbb{R}$  is an ambient charge density and, by definition,  $\bar{W}(r(t)) := \lim_{r \rightarrow r(t)} W(r, t)$ , with  $r(t) \in \mathbb{E}^3$  being the position of the string element with a proper rest reference coordinates  $(\tau, \sigma) \in \mathbb{E}^2$  at the time moment  $t = t(\tau, \sigma) \in \mathbb{R}$ .

We proceed now to constructing the dynamical Euler equations for our string model, making use of the general action functional (5.6) in the following form:

$$(5.10) \quad S^{(\tau)} = - \int_{\tau_1}^{\tau_2} d\tau \int_{\sigma_1(\tau)}^{\sigma_2(\tau)} \bar{W} |r'| (1 + |\dot{r}|^2 - \langle r', r' \rangle_{\mathbb{E}^3}^{-1}, \dot{r} \rangle_{\mathbb{E}^3}^2)^{1/2} d\sigma,$$

It is easy to calculate that the generalized momentum density

$$(5.11) \quad \begin{aligned} p & : = \partial \mathcal{L}^{(\tau)} / \partial \dot{r} = \frac{-\bar{W} |r'| (\dot{r} - r' |r'|^{-2} \langle \dot{r}, r' \rangle_{\mathbb{E}^3})}{(|\dot{r}|^2 + 1 - \langle r', r' \rangle_{\mathbb{E}^3}^{-1}, \dot{r} \rangle_{\mathbb{E}^3}^2)^{1/2}} = \\ & = \frac{-\bar{W} |r'| \hat{N} dr}{d\tau (|\dot{r}|^2 + 1 - \langle r', r' \rangle_{\mathbb{E}^3}^{-1}, \dot{r} \rangle_{\mathbb{E}^3}^2)^{1/2}} = -|r'| \bar{W} \hat{N} dr / dt = -|r'| \hat{N} (\bar{W} u) \end{aligned}$$

satisfies the dynamical equation

$$(5.12) \quad dp/d\tau \quad = \delta\mathcal{L}^{(\tau)}/\delta r = -(|r'|^2(|\dot{r}|^2 + 1) - \langle r', \dot{r} \rangle_{\mathbb{E}^3}^2)^{1/2} \nabla \bar{W} + \\ + \frac{\partial}{\partial \sigma} \left\{ \frac{\bar{W}(1 + |\dot{r}|^2 \hat{T}) r' / r'}{(1 + \langle \dot{r}|^2 \hat{T} r' / r'|^{-1}, r' / r'|^{-1} \rangle_{\mathbb{E}^3})^{1/2}} \right\},$$

where we denoted by

$$(5.13) \quad \mathcal{L}^{(\tau)} := -\bar{W}(|r'|^2(1 + |\dot{r}|^2) - \langle \dot{r}, r' \rangle_{\mathbb{E}^3}^2)^{1/2} = -\bar{W}(|r'|^2 + |\dot{r}|^2 \langle r', \hat{T} r' \rangle_{\mathbb{E}^3})^{1/2}$$

the corresponding Lagrangian function, and for any vector  $w \in \mathbb{E}^3$

$$(5.14) \quad \hat{T}_w := 1 - |w|^{-2} w \otimes w, \quad |w|^2 := \langle w, w \rangle_{\mathbb{E}^3}^2$$

the usual projector operator in  $\mathbb{E}^3$ . As a result of (5.12) one finds that

$$(5.15) \quad dp/dt \quad = -|r'| \nabla \bar{W} + \\ + (1 - |u|^2) \langle u, r' \rangle_{\mathbb{E}^3}^2)^{-1/2} \frac{\partial}{\partial \sigma} \left\{ \frac{\bar{W}(1 - |u|^2 + \langle u, r' \rangle_{\mathbb{E}^3}^2 + |u|^2 \hat{T}_u) r' / r'}{(1 - |u|^2 + \langle u, r' \rangle_{\mathbb{E}^3}^2)^{1/2}} \right\},$$

where we took into account that owing to (5.4)

$$(5.16) \quad \dot{r} = dr/d\tau = dr/dt \cdot (dt/d\tau) = u(1 - |u|^2 + \langle u, r' \rangle_{\mathbb{E}^3}^2)^{-1/2}.$$

The Lagrangian function is degenerate [10], satisfying the obvious identity

$$(5.17) \quad \langle p, r' \rangle_{\mathbb{E}^3} = 0$$

for all  $\tau \in \mathbb{R}$ . Concerning the Hamiltonian formulation of the dynamics (5.12) we construct the corresponding Hamiltonian functional as

$$(5.18) \quad \mathcal{H} := \int_{\sigma_1}^{\sigma_2} (\langle p, \dot{r} \rangle_{\mathbb{E}^3} - \mathcal{L}^{(\tau)}) d\sigma = \\ = \int_{\sigma_1}^{\sigma_2} \left( \frac{-\bar{W}|r'|(|\dot{r}|^2 - r'|r'|^{-2} \langle r', \dot{r} \rangle_{\mathbb{E}^3}^2)}{(|\dot{r}|^2 + 1 - \langle r'|r'|^{-1}, \dot{r} \rangle_{\mathbb{E}^3}^2)^{1/2}} + \frac{\bar{W}|r'|(|\dot{r}|^2 + 1 - \langle r'|r'|^{-1}, \dot{r} \rangle_{\mathbb{E}^3}^2)}{(|\dot{r}|^2 + 1 - \langle r'|r'|^{-1}, \dot{r} \rangle_{\mathbb{E}^3}^2)^{1/2}} \right) d\sigma = \\ = \int_{\sigma_1}^{\sigma_2} \left( \frac{\bar{W}|r'|}{(|\dot{r}|^2 + 1 - \langle r'|r'|^{-1}, \dot{r} \rangle_{\mathbb{E}^3}^2)^{1/2}} \right) d\sigma = - \int_{\sigma_1}^{\sigma_2} (\bar{W}^2 |r'|^2 - |p|^2)^{1/2} d\sigma,$$

satisfying the canonical Hamiltonian equations

$$(5.19) \quad dr/d\tau := \delta\mathcal{H}/\delta p, \quad dp/d\tau := -\delta\mathcal{H}/\delta r,$$

where

$$(5.20) \quad d\mathcal{H}/d\tau = 0,$$

holding only with respect to the proper rest reference frame  $\mathcal{K}_\tau$  time parameter  $\tau \in \mathbb{R}$ . Now making use of identity (5.17) the Hamiltonian functional (5.18) can be equivalently represented [10] in the symbolic form as

$$(5.21) \quad \mathcal{H} = \int_{\sigma_1}^{\sigma_2} |\bar{W} r' \pm ip|_{\mathbb{C}} d\sigma,$$

where  $i := \sqrt{-1}$ . Moreover, concerning the result obtained above we need to mention here that one can not construct the suitable Hamiltonian function expression and relationship of type (5.20) with respect to the laboratory reference frame  $\mathcal{K}$ , since expression (5.21) is not defined on the whole for a separate laboratory time parameter  $t \in \mathbb{R}$  locally dependent both on the spatial parameter  $\sigma \in \mathbb{R}$  and the proper rest reference frame time parameter  $\tau \in \mathbb{R}$ .

Thereby, one can formulate the following proposition.

**Proposition 5.1.** *The hadronic string model (5.1) allows, on the related world surface, the conformal local coordinates, with respect to which the resulting dynamics is described by means of the linear second order elliptic equation (5.7). Subject to the proper Euclidean rest reference frame  $\mathcal{K}_\tau$  coordinates the corresponding dynamics is equivalent to the canonical Hamiltonian equations (5.19) with Hamiltonian functional (5.18).*

We proceed now to construct the action functional expression for a charged string under an external electromagnetic field, generated by a point velocity charged particle  $\xi_f$ , moving with some velocity  $u_f := dr_f/dt \in \mathbb{E}^3$  subject to a laboratory reference frame  $\mathcal{K}$ . To solve this problem we make use of the trick of Section 3.1, passing to the string, considered with respect to the proper rest reference frame  $\mathcal{K}_r$  moving under the external vacuum field potential  $\bar{W}'$  with respect to the relative reference frame  $\mathcal{K}'_f$ , specified by its own Euclidean coordinates  $(t', r_f) \in \mathbb{E}^4$ , which simultaneously moves with velocity  $u_f \in \mathbb{E}^3$  with respect to the laboratory reference frame  $\mathcal{K}$ . As a result of this reasoning we can write down the action functional:

$$(5.22) \quad S^{(\tau)} = - \int_{\tau_1}^{\tau_2} d\tau \int_{\sigma_1(\tau)}^{\sigma_2(\tau)} \bar{W}'(|r'|^2(1 + |\dot{r} - \dot{r}_f|^2) - \langle \dot{r} - \dot{r}_f, r' \rangle_{\mathbb{E}^3}^2)^{1/2} d\sigma,$$

giving rise to the following dynamical equation

$$(5.23) \quad dP/d\tau : = \delta \mathcal{L}^{(\tau)} / \delta r = -(|r'|^2(1 + |\dot{r} - \dot{r}_f|^2) - \langle \dot{r} - \dot{r}_f, r' \rangle_{\mathbb{E}^3}^2)^{1/2} \nabla \bar{W}' + \frac{\partial}{\partial \sigma} \left\{ \frac{\bar{W}'(1 + |\dot{r} - \dot{r}_f|^2 \hat{T}_{\dot{r} - \dot{r}_f}) r'}{(|r'|^2(1 + |\dot{r} - \dot{r}_f|^2) - \langle \dot{r} - \dot{r}_f, r' \rangle_{\mathbb{E}^3}^2)^{1/2}} \right\},$$

where the generalized momentum density

$$(5.24) \quad P := \frac{-\bar{W}'|r'|^2 \hat{N}(\dot{r} - \dot{r}_f)}{(|r'|^2(1 + |\dot{r} - \dot{r}_f|^2) - \langle \dot{r} - \dot{r}_f, r' \rangle_{\mathbb{E}^3}^2)^{1/2}}.$$

Owing to (4.18), one can define

$$(5.25) \quad p : = \frac{-\bar{W}'|r'|^2 \hat{N}\dot{r}}{(|r'|^2(1 + |\dot{r} - \dot{r}_f|^2) - \langle \dot{r} - \dot{r}_f, r' \rangle_{\mathbb{E}^3}^2)^{1/2}} = -\frac{\bar{W}'|r'| \hat{N} dr}{d\tau(1 + |\dot{r} - \dot{r}_f|^2 - \langle \dot{r} - \dot{r}_f, r' / |r'| \rangle_{\mathbb{E}^3}^2)^{1/2}} = -\bar{W}'|r'| \hat{N} u' = -\bar{W}|r'| \hat{N} u$$

as the local string momentum density and

$$(5.26) \quad \xi |r'| A : = \frac{\bar{W}'|r'|^2 \hat{N} \dot{r}_f}{(|r'|^2(1 + |\dot{r} - \dot{r}_f|^2) - \langle \dot{r} - \dot{r}_f, r' \rangle_{\mathbb{E}^3}^2)^{1/2}} = \frac{\bar{W}'|r'| \hat{N} dr_f}{d\tau(1 + |\dot{r} - \dot{r}_f|^2 - \langle \dot{r} - \dot{r}_f, r' / |r'| \rangle_{\mathbb{E}^3}^2)^{1/2}} = \bar{W}'|r'| \hat{N} u'_f = \bar{W}|r'| \hat{N} u_f$$

as the external vector magnetic potential density, where  $\xi \in \mathbb{R}$  is a uniform charge density, distributed along the string length. Thus, equation (5.23) reduces to

$$(5.27) \quad d(p + \xi |r'| A) / dt' = -|r'| \nabla \bar{W}' + (1 - |u' - u'_f|^2 + \langle u' - u'_f, r' \rangle_{\mathbb{E}^3}^2)^{-1/2} \times \frac{\partial}{\partial \sigma} \left\{ \frac{\bar{W}'(1 - |u' - u'_f|^2 + \langle u' - u'_f, r' \rangle_{\mathbb{E}^3}^2 + |u' - u'_f|^2 \hat{T}_{u' - u'_f} r' / r')}{(1 - |u' - u'_f|^2 + \langle u' - u'_f, r' \rangle_{\mathbb{E}^3}^2)^{1/2}} \right\}$$

with respect to the moving reference frame  $\mathcal{K}'_f$ , or equivalently, to

$$(5.28) \quad d(p + \xi |r'| A) / dt = -|r'| \nabla \bar{W}(1 - |u_f|^2) + (1 - |u_f|^2)(1 - |u_f|^2 - |u - u_f|^2 + \langle u - u_f, r' \rangle_{\mathbb{E}^3}^2)^{-1/2} \times \frac{\partial}{\partial \sigma} \left\{ \frac{\bar{W}(1 - |u_f|^2 - |u - u_f|^2 + \langle u - u_f, r' \rangle_{\mathbb{E}^3}^2 + |u - u_f|^2 \hat{T}_{u - u_f} r' / r')}{(1 - |u_f|^2 - |u - u_f|^2 + \langle u - u_f, r' \rangle_{\mathbb{E}^3}^2)^{1/2}} \right\}$$

with respect to the moving laboratory frame  $\mathcal{K}$ . The latter can be easily rewritten also as the Lorentz type force expression

$$(5.29) \quad dp/dt = \xi |r'| E + \xi |r'| u \times B - \xi |r'| \nabla \langle u - u_f, A \rangle_{\mathbb{E}^3} + (1 - |u_f|^2)(1 - |u_f|^2 - |u - u_f|^2 + \langle u - u_f, r' \rangle_{\mathbb{E}^3}^2)^{-1/2} \times \frac{\partial}{\partial \sigma} \left\{ \frac{\bar{W}(1 - |u_f|^2 - |u - u_f|^2 + \langle u - u_f, r' \rangle_{\mathbb{E}^3}^2 + |u - u_f|^2 \hat{T}_{u - u_f} r' / r')}{(1 - |u_f|^2 - |u - u_f|^2 + \langle u - u_f, r' \rangle_{\mathbb{E}^3}^2)^{1/2}} \right\},$$

where  $B = \nabla \times A$  means, as usual, the external magnetic field and

$$(5.30) \quad E = \partial A / \partial t - \nabla \bar{W}$$

means the corresponding electric field, acting on the string. Making use of the standard scheme, one can derive, as above, the Hamiltonian interpretation of dynamical equations (5.23), but which will not be here discussed.

**5.3. Comments.** Based on the vacuum field theory approach, devised recently in [125, 126, 24], we revisited the alternative charged point particle and hadronic string electrodynamics models, having succeeded in treating their Lagrangian and Hamiltonian properties. The obtained results were compared with classical ones, owing to which a physically motivated choice of a true model was argued. Another important aspect of the developed vacuum field theory approach consists in singling out the decisive role of the related rest reference frame  $\mathcal{K}_r$ , with respect to which the relativistic object motion, in reality, realizes. Namely, with respect to the proper rest reference frame evolution parameter  $\tau \in \mathbb{R}$  all of our electrodynamics models allow both the Lagrangian and Hamiltonian physically reasonable formulations, suitable for the canonical procedure. The deeper physical nature of this fact remains, up today, as we assume, not enough understood. We would like to recall here only very interesting reasonings by R. Feynman, F.A. Berezin and M.S. Marinov, who argued in [57, 16] that the relativistic expressions obtain physical sense only with respect to the proper rest reference frames. In a sequel of our work we plan to analyze our relativistic electrodynamic models subject to their quantization and make a step toward the related vacuum quantum field theory of infinite many particle systems.

**5.4. Supplement: The Maxwell electromagnetism within the vacuum field theory.** We start from the following field theoretical model [19] of the microscopic vacuum medium structure, considered as some physical reality imbedded into the standard three-dimensional Euclidean space reference frame marked by three spatial coordinates  $r \in \mathbb{E}^3$ , endowed, as before, with the standard scalar product  $\langle \cdot, \cdot \rangle_{\mathbb{E}^3}$ , and parameterized by means of the scalar temporal parameter  $t \in \mathbb{R}$ . First we will describe the physical vacuum medium endowing it with an everywhere smooth enough four-vector potential function  $(W, A) : M^4 \rightarrow T^*(M^4)$ , defined in the Minkowski space  $M^4$  and naturally related to light propagation properties. The material objects, imbedded into the vacuum medium, we will model (classically here) by means of the scalar charge density function  $\rho : M^4 \rightarrow \mathbb{R}$  and the vector current density  $J : M^4 \rightarrow \mathbb{E}^3$ , being also everywhere smooth enough functions.

- (i) The *first* field theory principle regarding the vacuum we accept is formulated as follows: the four-vector function  $(W, A) : M^4 \rightarrow T^*(M^4)$  satisfies the standard Lorentz type continuity relationship

$$(5.31) \quad \frac{1}{c} \frac{\partial W}{\partial t} + \langle \nabla, A \rangle_{\mathbb{E}^3} = 0,$$

where, by definition,  $\nabla := \partial / \partial r$  is the usual gradient operator with respect to the spatial variable  $r \in \mathbb{E}^3$  and  $c$  is the light velocity in the vacuum.

- (ii) The *second* field theory principle we accept is a dynamical wave relationship on the scalar potential component  $W : M^4 \rightarrow \mathbb{R}$ :

$$(5.32) \quad \frac{1}{c^2} \frac{\partial^2 W}{\partial t^2} - \nabla^2 W = \rho,$$

assuming the linear law of the small vacuum uniform and isotropic perturbation propagations in the space-time, understood here, evidently, as a first (linear) approximation in the case of weak enough fields.

- (iii) The *third* principle is similar to the first one and means simply the continuity condition for the density and current density functions:

$$(5.33) \quad \partial \rho / \partial t + \langle \nabla, j \rangle_{\mathbb{E}^3} = 0.$$

We need to note here that the vacuum field perturbations velocity parameter  $c > 0$ , used above, coincides with the vacuum light velocity, as we are trying to derive successfully from these first

principles the well-known Maxwell electromagnetism field equations, to analyze the related Lorentz forces and special relativity relationships. To do this, we first combine equations (5.31) and (5.32):

$$\frac{1}{c^2} \frac{\partial^2 W}{\partial t^2} = - \langle \nabla, \frac{1}{c} \frac{\partial A}{\partial t} \rangle_{\mathbb{E}^3} = \langle \nabla, \nabla W \rangle_{\mathbb{E}^3} + \rho,$$

whence

$$(5.34) \quad \langle \nabla, -\frac{1}{c} \frac{\partial A}{\partial t} - \nabla W \rangle_{\mathbb{E}^3} = \rho.$$

Having put, by definition,

$$(5.35) \quad E := -\frac{1}{c} \frac{\partial A}{\partial t} - \nabla W,$$

we obtain the first material Maxwell equation

$$(5.36) \quad \langle \nabla, E \rangle_{\mathbb{E}^3} = \rho$$

for the electric field  $E : M^4 \rightarrow \mathbb{E}^3$ . Having now applied the rotor-operation  $\nabla \times$  to expression (5.35) we obtain the first Maxwell field equation

$$(5.37) \quad \frac{1}{c} \frac{\partial B}{\partial t} - \nabla \times E = 0$$

on the magnetic field vector function  $B : M^4 \rightarrow \mathbb{E}^3$ , defined as

$$(5.38) \quad B := \nabla \times A.$$

*Remark 5.2.* It is useful to remark that *the second field theory principle* is exactly equivalent to the experimentally stated physical relationships (5.35) and (5.36) for the electric field  $E : M^4 \rightarrow \mathbb{E}^3$ . Really, having applied the operator  $\nabla \times$  to the left-hand side of (5.35), one obtains the wave relationship (5.32).

To derive the second Maxwell field equation we will make use of (5.38), (5.31) and (5.35):

$$\begin{aligned} \nabla \times B &= \nabla \times (\nabla \times A) = \nabla \langle \nabla, A \rangle_{\mathbb{E}^3} - \nabla^2 A = \\ &= \nabla \left( -\frac{1}{c} \frac{\partial W}{\partial t} \right) - \nabla^2 A = \frac{1}{c} \frac{\partial}{\partial t} \left( -\nabla W - \frac{1}{c} \frac{\partial A}{\partial t} + \frac{1}{c} \frac{\partial A}{\partial t} \right) - \nabla^2 A = \\ (5.39) \quad &= \frac{1}{c} \frac{\partial E}{\partial t} + \left( \frac{1}{c^2} \frac{\partial^2 A}{\partial t^2} - \nabla^2 A \right). \end{aligned}$$

We have from (5.35), (5.36) and (5.33) that

$$\langle \nabla, \frac{1}{c} \frac{\partial E}{\partial t} \rangle_{\mathbb{E}^3} = \frac{1}{c} \frac{\partial \rho}{\partial t} = -\frac{1}{c} \langle \nabla, j \rangle_{\mathbb{E}^3},$$

or

$$(5.40) \quad \langle \nabla, -\frac{1}{c^2} \frac{\partial^2 A}{\partial t^2} - \nabla \left( \frac{1}{c} \frac{\partial W}{\partial t} \right) + \frac{1}{c} j \rangle_{\mathbb{E}^3} = 0.$$

Now making use of (5.31), from (5.40) we obtain that

$$(5.41) \quad \frac{1}{c^2} \frac{\partial^2 A}{\partial t^2} - \nabla^2 A = \frac{1}{c} (j + \nabla \times S)$$

for some smooth vector function  $S : M^4 \rightarrow \mathbb{E}^3$ . Here we need to note that continuity equation (5.33) is defined, concerning the current density vector  $j : M^4 \rightarrow \mathbb{R}^3$ , up to a vorticity expression, that is  $j \simeq j + \nabla \times S$  and equation (5.41) can finally be rewritten as

$$(5.42) \quad \frac{1}{c^2} \frac{\partial^2 A}{\partial t^2} - \nabla^2 A = \frac{1}{c} j.$$

Having substituted (5.42) into (5.39) we obtain the second Maxwell field equation

$$(5.43) \quad \nabla \times B - \frac{1}{c} \frac{\partial E}{\partial t} = \frac{1}{c} j.$$

In addition, from (5.38) one also finds the magnetic no-charge relationship

$$(5.44) \quad \langle \nabla, B \rangle_{\mathbb{E}^3} = 0.$$

Thus, we have derived all the Maxwell electromagnetic field equations from our three main principles (5.31), (5.32) and (5.33). The success of our undertaking will be more impressive if

we adapt our results to those following from the well-known relativity theory in the case of point charges or masses. Below we will try to demonstrate the corresponding derivations based on some completely new physical conceptions of the vacuum medium first discussed in [136, 24].

It is interesting to analyze a particular case of the first field theory vacuum principle (5.31) when the following local conservation law for the scalar potential field function  $W : M^4 \rightarrow \mathbb{R}$  holds:

$$(5.45) \quad \frac{d}{dt} \int_{\Omega_t} W(t, r') d^3 r' = 0,$$

where  $\Omega_t \subset \mathbb{E}^3$  is any open domain in space  $\mathbb{E}^3$  with the smooth boundary  $\partial\Omega_t$  for all  $t \in \mathbb{R}$  and  $d^3 r'$  is the standard volume measure in  $\mathbb{E}^3$  in a vicinity of the point  $r'(t) := r_f(t) - r \in \Omega_t$ . Having calculated expression (5.45) we obtain the following equivalent continuity equation

$$(5.46) \quad \frac{1}{c} \frac{\partial W}{\partial t} + \langle \nabla, \frac{u}{c} W \rangle_{\mathbb{E}^3} = 0,$$

where  $\nabla := \partial/\partial r'$  is, as above, the gradient operator with respect to the spatial variable  $r' \in \mathbb{E}^3$  and  $u := dr_f(t)/dt$  is the velocity vector of a vacuum medium perturbation influenced by the charge  $\xi$ , located at point  $r_f(t) \in \mathbb{E}^3$  and carrying the field potential quantity  $W$ . Comparing now equations (5.31), (5.46) and using equation (5.33) we can make the suitable very important identifications:

$$(5.47) \quad A = \frac{u}{c} W, \quad j = \rho u,$$

well known from the classical electrodynamics [97] and superconductivity theory [57, 87]. Thus, we are faced with a new physical interpretation of the conservative electromagnetic field theory when the vector potential  $A : M^4 \rightarrow \mathbb{E}^3$  is completely determined via expression (5.47) by the scalar field potential function  $W : M^4 \rightarrow \mathbb{R}$ . It is also evident that all the Maxwell electromagnetism field equations derived above hold as well in the case

Consider now the conservation equation (5.45) jointly with the related integral "vacuum momentum" conservation condition

$$(5.48) \quad \frac{d}{dt} \int_{\Omega_t} \left( \frac{Wv}{c^2} \right) d^3 r' = 0, \quad \Omega_t|_{t=0} = \Omega_0,$$

where, as above,  $\Omega_t \subset \mathbb{E}^3$  is for any time  $t \in \mathbb{R}$  an open domain with the smooth boundary  $\partial\Omega_t$ , whose evolution is governed by the equation

$$(5.49) \quad dr'/dt = u(r', t)$$

for all  $r' \in \Omega_t$  and  $t \in \mathbb{R}$ , as well as by the initial state of the boundary  $\partial\Omega_0$ . As a result of relation (5.48) one obtains the new continuity equation

$$(5.50) \quad \frac{d(uW)}{dt} + uW \langle \nabla, u \rangle_{\mathbb{E}^3} = 0.$$

Now making use of (5.46) in the equivalent form

$$\frac{dW}{dt} + W \langle \nabla, u \rangle_{\mathbb{E}^3} = 0,$$

we finally obtain a very interesting local conservation relationship

$$(5.51) \quad du/dt = 0$$

on the vacuum matter perturbations velocity  $u = dr'/dt$ , which holds for all values of the time parameter  $t \in \mathbb{R}$ . As it is easy to observe, the obtained relationship completely coincides with the well-known hydrodynamic equation [103] of ideal compressible liquid without any external exertion, that is, any external forces and field "pressure" are equally identical to zero. We received a natural enough result where the propagation velocity of the vacuum field matter is constant and equals exactly  $u = c$ , that is the light velocity in the vacuum, if to take into account the starting wave equation (5.32) owing to which the small vacuum field matter perturbations propagate in the space with the light velocity.



## 6. THE ELECTROMAGNETIC DIRAC-FOCK-PODOLSKY PROBLEM AND SYMPLECTIC PROPERTIES OF THE MAXWELL AND YANG-MILLS TYPE DYNAMICAL SYSTEMS

**6.1. Introduction.** When investigating different dynamical systems on canonical symplectic manifolds, invariant under action of certain symmetry groups, additional mathematical structures often appear, the analysis of which shows their importance for understanding many related problems under study. Amongst them we here mention the Cartan type connection on an associated principal fiber bundle, which enables one to study in more detail the properties of the investigated dynamical system in the case of its reduction upon the corresponding invariant submanifolds and quotient spaces, associated with them.

Problems related to the investigation of properties of reduced dynamical systems on symplectic manifolds were studied, e.g., in [1, 119, 20, 132, 131], where the relationship between a symplectic structure on the reduced space and the available connection on a principal fiber bundle was formulated in explicit form. Other aspects of dynamical systems related to properties of reduced symplectic structures were studied in [95, 135, 75, 76], where, in particular, the reduced symplectic structure was explicitly described within the framework of the classical Dirac scheme, and several applications to nonlinear (including celestial) dynamics were given.

It is well-known [28, 148, 40, 125, 24, 116] that the Hamiltonian theory of electromagnetic Maxwell equations faces a very important classical problem of introducing into the unique formalism the well known Lorenz conditions, ensuring both the wave structure of propagating quanta and the positivity of energy. Regretfully, in spite of classical studies on this problem given by Dirac, Fock and Podolsky [43], the problem remains open, and the Lorenz condition is imposed within the modern electrodynamics as the external constraint not entering a priori the initial Hamiltonian (or Lagrangian) theory. Moreover, when trying to quantize the electromagnetic theory, as it was shown by Pauli, Dirac, Bogolubov and Shirkov and others [28, 116, 40, 18], within the existing approaches the quantum Lorenz condition could not be satisfied, except in the average sense, since it becomes not compatible with the related quantum dynamics. This problem stimulated us to study this problem from the so called symplectic reduction theory [104, 131, 132], which allows the systematic introduction into the Hamiltonian formalism the external charge and current conditions, giving rise to a partial solution to the Lorenz condition problem mentioned above. Some applications of the method to Yang-Mills type equations interacting with a point charged particle, are presented in detail. In particular, based on analysis of reduced geometric structures on fibered manifolds, invariant under the action of a symmetry group, we construct the symplectic structures associated with connection forms on suitable principal fiber bundles. We present suitable mathematical preliminaries of the related Poissonian structures on the corresponding reduced symplectic manifolds, which are often used [1, 104, 96] in various problems of dynamics in modern mathematical physics, and apply them to study the non-standard Hamiltonian properties of the Maxwell and Yang-Mills type dynamical systems. We formulate a symplectic analysis of the important Lorenz type constraints, which describe the electrodynamic vacuum properties.

We formulate a symplectic reduction theory of the classical Maxwell electromagnetic field equations and prove [27] that the important Lorenz condition, ensuring the existence of electromagnetic waves [28, 18, 57, 97], can be naturally included into the Hamiltonian picture, thereby solving the Dirac, Fock and Podolsky problem [43] mentioned above. We also study from the symplectic reduction theory the Poissonian structures and the classical minimal interaction principle related with Yang-Mills type equations.

**6.2. The symplectic reduction on cotangent fiber bundles with symmetry.** Consider an  $n$ -dimensional smooth manifold  $M$  and the cotangent vector fiber bundle  $T^*(M)$ . We equip (see [64], Chapter VII; [47]) the cotangent space  $T^*(M)$  with the canonical Liouville 1-form  $\lambda(\alpha^{(1)}) := pr_M^* \alpha^{(1)} \in \Lambda^1(T^*(M))$ , where  $pr_M : T^*(M) \rightarrow M$  is the canonical projection and, by definition,

$$(6.1) \quad \alpha^{(1)}(u) = \sum_{j=1}^n v_j du^j,$$

where  $(u, v) \in T^*(M)$  are the corresponding canonical local coordinates on  $T^*(M)$ . Thus, any group of diffeomorphisms of the manifold  $M$  naturally lifted to the fiber bundle  $T^*(M)$  preserves the invariance of the canonical 1-form  $\lambda(\alpha^{(1)}) \in \Lambda^1(T^*(M))$ . In particular, if a smooth action of a Lie group  $G$  is given on the manifold  $M$ , then every element  $a \in \mathcal{G}$ , where  $\mathcal{G}$  is the Lie algebra

of the Lie group  $G$ , generates the vector field  $k_a \in T(M)$  in a natural manner. Furthermore, since the group action on  $M$ , i.e.,

$$(6.2) \quad \varphi : G \times M \rightarrow M,$$

generates a diffeomorphism  $\varphi_g \in \text{Diff } M$  for every element  $g \in G$ , this diffeomorphism is naturally lifted to the corresponding diffeomorphism  $\varphi_g^* \in \text{Diff } T^*(M)$  of the cotangent fiber bundle  $T^*(M)$ , which also leaves the canonical 1-form  $pr_M^* \alpha^{(1)} \in \Lambda^1(T^*(M))$  invariant. Namely, the equality

$$(6.3) \quad \varphi_g^* \lambda(\alpha^{(1)}) = \lambda(\alpha^{(1)})$$

holds [1, 64, 119] for every 1-form  $\alpha^{(1)} \in \Lambda^1(M)$ . Thus, we can define on  $T^*(M)$  the corresponding vector field  $K_a : T^*(M) \rightarrow T(T^*(M))$  for every element  $a \in \mathcal{G}$ . Then condition (6.3) can be rewritten in the following form for all  $a \in \mathcal{G}$ :

$$L_{K_a} \cdot pr_M^* \alpha^{(1)} = pr_M^* \cdot L_{k_a} \alpha^{(1)} = 0,$$

where  $L_{K_a}$  and  $L_{k_a}$  are the ordinary Lie derivatives on  $\Lambda^1(T^*(M))$  and  $\Lambda^1(M)$ , respectively.

The canonical symplectic structure on  $T^*(M)$  is defined as

$$(6.4) \quad \omega^{(2)} := d\lambda(\alpha^{(1)})$$

and is invariant, i.e.,  $L_{K_a} \omega^{(2)} = 0$  for all  $a \in \mathcal{G}$ .

For any smooth function  $H \in D(T^*(M))$ , a Hamiltonian vector field  $K_H : T^*(M) \rightarrow T(T^*(M))$  such that

$$(6.5) \quad i_{K_H} \omega^{(2)} = -dH$$

is defined, and vice versa, because the symplectic 2-form (6.4) is non-degenerate. Using (6.5) and (6.4), we easily establish that the Hamiltonian function  $H := H_K \in D(T^*(M))$  is given by the expression  $H_K = pr_M^* \alpha^{(1)}(K_H) = \alpha^{(1)}(pr_M^* K_H) = \alpha^{(1)}(k_H)$ , where  $k_H \in T(M)$  is the corresponding vector field on the manifold  $M$ , whose lifting to the fiber bundle  $T^*(M)$  coincides with the vector field  $K_H : T^*(M) \rightarrow T(T^*(M))$ . For  $K_a : T^*(M) \rightarrow T(T^*(M))$ , where  $a \in \mathcal{G}$ , it is easy to establish that the corresponding Hamiltonian function  $H_a = \alpha^{(1)}(k_a) = pr_M^* \alpha^{(1)}(K_a)$  for  $a \in \mathcal{G}$  defines [1, 119, 20] a linear momentum mapping  $l : T^*(M) \rightarrow \mathcal{G}^*$  according to the rule

$$(6.6) \quad H_a := \langle l, a \rangle,$$

where  $\langle \cdot, \cdot \rangle$  is the corresponding convolution on  $\mathcal{G}^* \times \mathcal{G}$ . By virtue of definition (6.6), the momentum mapping  $l : T^*(M) \rightarrow \mathcal{G}^*$  is invariant under the action of any invariant Hamiltonian vector field  $K_b : T^*(M) \rightarrow T(T^*(M))$  for any  $b \in \mathcal{G}$ . Indeed,  $L_{K_b} \langle l, a \rangle = L_{K_b} H_a = -L_{K_a} H_b = 0$ , because, by definition, the Hamiltonian function  $H_b \in D(T^*(M))$  is invariant under the action of any vector field  $K_a : T^*(M) \rightarrow T(T^*(M))$ ,  $a \in \mathcal{G}$ .

We now fix a regular value of the momentum mapping  $l(u, v) = \xi \in \mathcal{G}^*$  and consider the corresponding submanifold  $\mathcal{M}_\xi := \{(u, v) \in T^*(M) : l(u, v) = \xi \in \mathcal{G}^*\}$ . On the basis of definition (6.1) and the invariance of the 1-form  $pr_M^* \alpha^{(1)} \in \Lambda^1(T^*(M))$  under the action of the Lie group  $G$  on  $T^*(M)$ , we can write the equalities

$$(6.7) \quad \begin{aligned} & \langle l(g \circ (u, v)), a \rangle = pr_M^* \alpha^{(1)}(K_a)(g \circ (u, v)) = \\ & = pr_M^* \alpha^{(1)}(K_{Ad_{g^{-1}}a})(u, v) := \\ & = \langle l(u, v), Ad_{g^{-1}}a \rangle = \langle Ad_{g^{-1}}^* l(u, v), a \rangle \end{aligned}$$

for any  $g \in G$  and all  $a \in \mathcal{G}$  and  $(u, v) \in T^*(M)$ . Using (6.7) we establish that, for every  $g \in G$  and all  $(u, v) \in T^*(M)$ , the following relation is true:  $l(g \circ (u, v)) = Ad_{g^{-1}}^* l(u, v)$ . This means that the diagram

$$\begin{array}{ccc} T^*(M) & \xrightarrow{l} & \mathcal{G}^* \\ g \downarrow & & \downarrow Ad_{g^{-1}}^* \\ T^*(M) & \xrightarrow{l} & \mathcal{G}^* \end{array}$$

is commutative for all elements  $g \in G$ ; the corresponding action  $g : T^*(M) \rightarrow T^*(M)$  is called equivariant [1, 119].

Let  $G_\xi \subset G$  denote the stabilizer of a regular element  $\xi \in \mathcal{G}^*$  with respect to the related coadjoint action. It is obvious that in this case the action of the Lie subgroup  $G_\xi$  on the submanifold  $\mathcal{M}_\xi \subset T^*(M)$  is naturally defined; we assume that it is free and proper. According to this action on  $\mathcal{M}_\xi$ , we can define [1, 135, 76, 110, 96] a so-called reduced space  $\bar{\mathcal{M}}_\xi$  by taking the factor with respect to the action of the subgroup  $G_\xi$  on  $\mathcal{M}_\xi$ , i.e.,

$$(6.8) \quad \bar{\mathcal{M}}_\xi := \mathcal{M}_\xi / G_\xi.$$

The quotient space (6.8) induces a symplectic structure  $\bar{\omega}_\xi^{(2)} \in \Lambda^2(\bar{\mathcal{M}}_\xi)$  on itself, which is defined as follows:

$$(6.9) \quad \bar{\omega}_\xi^{(2)}(\bar{\eta}_1, \bar{\eta}_2) = \omega_\xi^{(2)}(\eta_1, \eta_2),$$

where  $\bar{\eta}_1, \bar{\eta}_2 \in T(\bar{\mathcal{M}}_\xi)$  are arbitrary vectors onto which vectors  $\eta_1, \eta_2 \in T(\mathcal{M}_\xi)$  are projected, taken at any point  $(u_\xi, v_\xi) \in \mathcal{M}_\xi$ , being uniquely projected onto the point  $\bar{\mu}_\xi \in \bar{\mathcal{M}}_\xi$ , according to (6.8).

Let  $\pi_\xi : \mathcal{M}_\xi \rightarrow T^*(M)$  denote the corresponding imbedding mapping into  $T^*(M)$  and let  $r_\xi : \mathcal{M}_\xi \rightarrow \bar{\mathcal{M}}_\xi$  denote the corresponding reduction to the space  $\bar{\mathcal{M}}_\xi$ . Then relation (6.9) can be rewritten equivalently in the form of the equality

$$(6.10) \quad r_\xi^* \bar{\omega}_\xi^{(2)} = \pi_\xi^* \omega_\xi^{(2)},$$

defined on vectors on the cotangent space  $T^*(\mathcal{M}_\xi)$ . To establish the symplecticity of the 2-form  $\omega_\xi^{(2)} \in \Lambda^2(\bar{\mathcal{M}}_\xi)$ , we use the corresponding non-degeneracy of the Poisson bracket  $\{\cdot, \cdot\}_\xi^r$  on  $\bar{\mathcal{M}}_\xi$ . To calculate it, we use a Dirac type construction, defining functions on  $\bar{\mathcal{M}}_\xi$  as certain  $G_\xi$ -invariant functions on the submanifold  $\mathcal{M}_\xi$ . Then one can calculate the Poisson bracket  $\{\cdot, \cdot\}_\xi$  of such functions that corresponds to symplectic structure (6.4) as an ordinary Poisson bracket on  $T^*(M)$ , arbitrarily extending these functions from the submanifold  $\mathcal{M}_\xi \subset T^*(M)$  to a certain neighborhood  $U(\mathcal{M}_\xi) \subset T^*(M)$ . It is obvious that two extensions of a given function to the neighborhood  $U(\mathcal{M}_\xi)$  of this type differ by a function that vanishes on the submanifold  $\mathcal{M}_\xi \subset T^*(M)$ . The difference between the corresponding Hamiltonian fields of these two different extensions to  $U(\mathcal{M}_\xi)$  is completely controlled by the conditions of the following lemma (see also [1, 119, 75, 76, 135, 132]).

**Lemma 6.1.** *Suppose that a function  $f : U(\mathcal{M}_\xi) \rightarrow \mathbb{R}$  is smooth and vanishes on  $\mathcal{M}_\xi \subset T^*(M)$ , i.e.,  $f|_{\mathcal{M}_\xi} = 0$ . Then, at every point  $(u_\xi, v_\xi) \in \mathcal{M}_\xi$  the corresponding Hamiltonian vector field  $K_f \in T(U(\mathcal{M}_\xi))$  is tangent to the orbit  $Or(G; (u_\xi, v_\xi))$ .*

*Proof.* It is obvious that the submanifold  $\mathcal{M}_\xi \subset T^*(M)$  is defined by a certain collection of relations of the type

$$(6.11) \quad H_{a_s} = \xi_s, \quad \xi_s := \langle \xi, a_s \rangle,$$

where  $a_s \in \mathcal{G}$ ,  $s = \overline{1, \dim \mathcal{G}}$ , is a certain basis of the Lie algebra  $\mathcal{G}$ , which follows from definition (6.6). Since a function  $f : U(\mathcal{M}_\xi) \rightarrow \mathbb{R}$  vanishes on  $\mathcal{M}_\xi$ , we can write the following equality:

$$f = \sum_{s=1}^{\dim \mathcal{G}} (H_{a_s} - \xi_s) f_s,$$

where  $f_s : U(\mathcal{M}_\xi) \rightarrow \mathbb{R}$ ,  $s = \overline{1, \dim \mathcal{G}}$ , is a certain collection of functions in the neighborhood  $U(\mathcal{M}_\xi)$ . We take an arbitrary tangent vector  $\eta \in T(U(\mathcal{M}_\xi))$  at the point  $(u_\xi, v_\xi) \in \mathcal{M}_\xi$  and

calculate the expression

$$\begin{aligned}
(6.12) \quad & \langle df(u_\xi, v_\xi), \eta(u_\xi, v_\xi) \rangle = \sum_{s=1}^{\dim \mathcal{G}} \langle dH_{a_s}(u_\xi, v_\xi), \eta(u_\xi, v_\xi) \rangle f_s(u_\xi, v_\xi) = \\
& = - \sum_{s=1}^{\dim \mathcal{G}} \omega^{(2)}(K_{a_s}(u_\xi, v_\xi), \eta(u_\xi, v_\xi)) f_s(u_\xi, v_\xi) = \\
& = -\omega^{(2)}\left(\sum_{s=1}^{\dim \mathcal{G}} K_{a_s}(u_\xi, v_\xi) f_s(u_\xi, v_\xi), \eta(u_\xi, v_\xi)\right) = \\
& = - \langle i\left(\sum_{s=1}^{\dim \mathcal{G}} K_{a_s}(u_\xi, v_\xi) f_s(u_\xi, v_\xi)\right) \omega^{(2)}, \eta(u_\xi, v_\xi) \rangle.
\end{aligned}$$

It follows from the arbitrariness of the vector  $\eta \in T(\mathcal{M}_\xi)$  at the point  $(u_\xi, v_\xi) \in \mathcal{M}_\xi$  and relation (6.12) that

$$K_f = \sum_{s=1}^{\dim \mathcal{G}} K_{a_s} f_s,$$

i.e.,  $K_f : \mathcal{M}_\xi \rightarrow T(Or(G))$ , which was to be proved.  $\square$

As a corollary of Lemma 6.1, we obtain an algorithm for the determination of the reduced Poisson bracket  $\{\cdot, \cdot\}_\xi^r$  on the space  $\bar{\mathcal{M}}_\xi$  according to definition (6.10). Namely, we choose two functions defined on  $\mathcal{M}_\xi$  and invariant under the action of the subgroup  $G_\xi$  and arbitrarily smoothly extend them to a certain open domain  $U(\mathcal{M}_\xi) \subset T^*(M)$ . Then we determine the corresponding Hamiltonian vector fields on  $T^*(M)$  and project them onto the space tangent to  $\mathcal{M}_\xi$ , adding, if necessary, the corresponding vectors tangent to the orbit  $Or(G)$ . It is obvious that the projections obtained depend on the chosen extensions to the domain  $U(\mathcal{M}_\xi) \subset T^*(M)$ . As a result, we establish that the reduced Poisson bracket  $\{\cdot, \cdot\}_\xi^r$  is uniquely defined via the restriction of the initial Poisson bracket upon  $\mathcal{M}_\xi \subset T^*(M)$ . By virtue of the non-degeneracy of the latter and the functional independence of the basis functions (6.11) on the submanifold  $U(\mathcal{M}_\xi) \subset T^*(M)$ , the reduced Poisson bracket  $\{\cdot, \cdot\}_\xi^r$  appears to be [1, 119, 135] non-degenerate on  $\bar{\mathcal{M}}_\xi$ . As a consequence of the non-degeneracy, we establish that the dimension of the reduced space  $\bar{\mathcal{M}}_\xi$  is even. Taking into account that the element  $\xi \in \mathcal{G}^*$  is regular and the dimension of the Lie algebra of the stabilizer  $\mathcal{G}_\xi$  is equal to  $\dim G_\xi$ , we easily establish that  $\dim \bar{\mathcal{M}}_\xi = \dim T^*(M) - 2\dim \mathcal{G}_\xi$ . Since, by construction,  $\dim T^*(M) = 2n$ , we conclude that the dimension of the reduced space  $\bar{\mathcal{M}}_\xi$  is necessarily even.

For the correctness of the algorithm, it is necessary to establish the existence of the corresponding projections of Hamiltonian vector fields onto the tangent space  $T(\mathcal{M}_\xi)$ . The following statement is true.

**Theorem 6.2.** *At every point  $(u_\xi, v_\xi) \in \mathcal{M}_\xi$ , one can choose a vector  $V_f \in T(Or(G))$  such that  $K_f(u_\xi, v_\xi) + V_f(u_\xi, v_\xi) \in T_{(u_\xi, v_\xi)}(\mathcal{M}_\xi)$ . Furthermore, the vector  $V_f \in T(Or(G))$  is determined uniquely up to a vector tangent to the orbit  $Or(G_\xi)$ .*

*Proof.* Note that the orbit  $Or(G; (u_\xi, v_\xi))$  passing through the point  $(u_\xi, v_\xi) \in \mathcal{M}_\xi$  is always symplectically orthogonal to the tangent space  $T_{(u_\xi, v_\xi)}(\mathcal{M}_\xi)$ . Indeed, for any vector  $\eta \in T(\mathcal{M}_\xi)$  and  $a \in \mathcal{G}$ , we have  $\omega^{(2)}(\eta, K_a) = -i_{K_a}\omega^{(2)}(\eta) = dH_a(\eta) = 0$ , because the submanifold  $\mathcal{M}_\xi \subset T^*(M)$  is defined by the equality  $\langle \xi, a \rangle = H_a$  for all  $a \in \mathcal{G}$ , i.e.,  $dH_a = 0$  on  $\mathcal{M}_\xi$ . Thus,  $T(\mathcal{M}_\xi) \cap T(Or(G)) = T(Or(G))$  because  $H_a \circ g_\xi = H_a$  for all  $g_\xi \in G_\xi$ , which follows from the invariance of the element  $\xi \in \mathcal{G}^*$  under the action of the Lie group  $G_\xi$ . We now solve the imbedding

condition  $K_f + V_f \in T(\mathcal{M}_\xi)$ , or the equation

$$(6.13) \quad \omega^{(2)}(K_f + V_f, K_a) = 0$$

on the manifold  $\mathcal{M}_\xi \subset T^*(M)$  for all  $a \in \mathcal{G}$ . We rewrite equality (6.13) in the form

$$(6.14) \quad K_a f = \omega^{(2)}(V_f, K_a)$$

on  $\mathcal{M}_\xi$  for all  $a \in \mathcal{G}$ ; it is obvious that the 2-form on the right-hand side of (6.14) depends only on the element  $\xi \in \mathcal{G}^*$ . Taking into account the equivariance of the group action on  $T^*(M)$  and the obvious equality

$$\omega^{(2)}(K_a, K_b) = pr_M^* \alpha^{(1)}([K_a, K_b]) = -pr_M^* \alpha^{(1)}(K_{[a, b]})$$

for all  $a, b \in \mathcal{G}$ , we establish that there exists an element  $a_f \in \mathcal{G}$  such that  $V_f = K_{a_f} \in T(Or(G))$  and

$$\begin{aligned}
 \omega^{(2)}(V_f, K_a) &= \omega^{(2)}(K_{a_f}, K_a) = pr_M^* \alpha^{(1)}([K_a, K_{a_f}]) = \\
 &= pr_M^* \alpha^{(1)}(K_{[a_f, a]}) = H_{[a_f, a]} = \langle l, [a_f, a] \rangle = \\
 (6.15) \quad &= \langle \xi, [a_f, a] \rangle = \langle ad_{a_f}^* \xi, a \rangle
 \end{aligned}$$

on  $\mathcal{M}_\xi$  for all  $a \in \mathcal{G}$ . Since  $ad_{a_f}^* \xi = 0$  for any  $a_f \in \mathcal{G}_\xi$ , we conclude that, on the quotient space  $\mathcal{G}/\mathcal{G}_\xi$  the right-hand side of (6.15) defines a non-degenerate skew-symmetric form associated with the canonical isomorphism  $\hat{\xi} : \mathcal{G}/\mathcal{G}_\xi \rightarrow (\mathcal{G}/\mathcal{G}_\xi)^*$ , where, by definition,

$$(6.16) \quad \langle \hat{\xi}(\tilde{a}), \tilde{b} \rangle := \langle \xi, [a, b] \rangle$$

for any  $\tilde{a}$  and  $\tilde{b} \in \mathcal{G}/\mathcal{G}_\xi$  with the corresponding representatives  $a$  and  $b \in \mathcal{G}$ . Further, since the function  $f : \mathcal{M}_\xi \rightarrow \mathbb{R}$  is  $G_\xi$ -invariant on  $\mathcal{M}_\xi \subset T^*(M)$ , the right-hand side of (6.14) defines an element  $\mu_f \in (\mathcal{G}/\mathcal{G}_\xi)^*$  by the equality

$$\mu_f : \tilde{a} := -K_a f$$

for all  $a \in G$ . Using relations (6.15) and (6.16), we establish that there exists the element

$$\tilde{a}_f = \hat{\xi}^{-1} \circ \mu_f \in \mathcal{G}/\mathcal{G}_\xi.$$

Since the element  $\tilde{a}_f \in \mathcal{G}/\mathcal{G}_\xi$  is associated with the element  $a_f \pmod{\mathcal{G}_\xi} \in \mathcal{G}$ , which uniquely generates a locally defined vector field  $K_{a_f} : Or(G) \rightarrow T(Or(G))$ , using the fact that  $V_f = K_{a_f}$  on  $\mathcal{M}_\xi$ , we complete the proof of the theorem.  $\square$

Now assume that two functions  $f_1, f_2 \in D(\mathcal{M}_\xi)$  are  $G_\xi$ -invariant. Then their reduced Poisson bracket  $\{f_1, f_2\}_\xi^r$  on  $\bar{\mathcal{M}}_\xi$  is defined according to the rule:

$$(6.17) \quad \{f_1, f_2\}_\xi^r := -\omega^{(2)}(K_{f_1} + V_{f_1}, K_{f_2} + V_{f_2}) = \{f_1, f_2\} + \omega^{(2)}(V_{f_1}, V_{f_2}),$$

where we have used the following identities on  $\mathcal{M}_\xi \subset T^*(M)$ :

$$\omega^{(2)}(K_{f_1} + V_{f_1}, V_{f_2}) = 0 = \omega^{(2)}(K_{f_2} + V_{f_2}, V_{f_1}),$$

being simple consequences of equality (6.13) on  $\mathcal{M}_\xi$ . Regarding (6.15), relation (6.17) takes the form

$$(6.18) \quad \{f_1, f_2\}_\xi^r = \{f_1, f_2\} + \frac{1}{2}(V_{f_1} f_2 - V_{f_2} f_1),$$

where  $f_1, f_2 \in D(\mathcal{M}_\xi)$  are arbitrary smooth extensions of the  $G_\xi$ -invariant functions defined earlier on the domain  $U(\mathcal{M}_\xi)$ . Thus, the following theorem holds.

**Theorem 6.3.** *The reduced Poisson bracket of two functions on the quotient space  $\bar{\mathcal{M}}_\xi = \mathcal{M}_\xi/G_\xi$  is determined with the use of their arbitrary smooth extensions to functions on an open neighborhood  $U(\mathcal{M}_\xi)$  according to the Dirac-type formula (6.18).*

**6.2.1. The symplectic reduction on principal fiber bundles with connection.** We begin by reviewing the backgrounds of the reduction theory subject to Hamiltonian systems with symmetry on principle fiber bundles. The material is partly available in [62, 95], so here it will be only sketched in notations suitable for us.

Let  $G$  denote a given Lie group with the unity element  $e \in G$  and the corresponding Lie algebra  $\mathcal{G} \simeq T_e(G)$ . Consider a principal fiber bundle  $p : (M, \varphi) \rightarrow N$  with the structure group  $G$  and base manifold  $N$ , on which the Lie group  $G$  acts by means of a mapping  $\varphi : M \times G \rightarrow M$ . Namely, for each  $g \in G$  there is a group diffeomorphism  $\varphi_g : M \rightarrow M$ , generating for any fixed  $u \in M$  the following induced mapping:  $\hat{u} : G \rightarrow M$ , where

$$(6.19) \quad \hat{u}(g) = \varphi_g(u).$$

On the principal fiber bundle  $p : (M, \varphi) \rightarrow N$  a connection  $\Gamma(\mathcal{A})$  is assigned by means of such a morphism  $\mathcal{A} : (T(M), \varphi_{g*}) \rightarrow (\mathcal{G}, Ad_{g^{-1}})$  that for each  $u \in M$  a mapping  $\mathcal{A}(u) : T_u(M) \rightarrow \mathcal{G}$  is a left inverse one to the mapping  $\hat{u}_*(e) : \mathcal{G} \rightarrow T_u(M)$ , that is

$$(6.20) \quad \mathcal{A}(u)\hat{u}_*(e) = 1.$$

As usual, denote by  $\varphi_g^* : T^*(M) \rightarrow T^*(M)$  the corresponding lift of the mapping  $\varphi_g : M \rightarrow M$  at any  $g \in G$ . If  $\alpha^{(1)} \in \Lambda^1(M)$  is the canonical  $G$ -invariant 1-form on  $M$ , the canonical symplectic structure  $\omega^{(2)} \in \Lambda^2(T^*(M))$  given by

$$(6.21) \quad \omega^{(2)} := d \, pr^* \alpha^{(1)}$$

generates the corresponding momentum mapping  $l : T^*(M) \rightarrow \mathcal{G}^*$ , where

$$(6.22) \quad l(\alpha^{(1)})(u) = \hat{u}^*(e)\alpha^{(1)}(u)$$

for all  $u \in M$ . Remark here that the principal fiber bundle structure  $p : (M, \varphi) \rightarrow N$  means in part the exactness of the following sequences of mappings:

$$(6.23) \quad 0 \rightarrow \mathcal{G} \xrightarrow{\hat{u}_*(e)} T_u(M) \xrightarrow{p^*(u)} T_{p(u)}(N) \rightarrow 0,$$

that is

$$(6.24) \quad p_*(u)\hat{u}_*(e) = 0 = \hat{u}^*(e)p^*(u)$$

for all  $u \in M$ . Combining (6.24) with (6.20) and (6.22), one obtains such an embedding:

$$(6.25) \quad [1 - \mathcal{A}^*(u)\hat{u}^*(e)]\alpha^{(1)}(u) \in \text{range } p^*(u)$$

for the canonical 1-form  $\alpha^{(1)} \in \Lambda^1(M)$  at  $u \in M$ . The expression (6.25) means of course, that

$$(6.26) \quad \hat{u}^*(e)[1 - \mathcal{A}^*(u)\hat{u}^*(e)]\alpha^{(1)}(u) = 0$$

for all  $u \in M$ . Now taking into account that the mapping  $p^*(u) : T^*(N) \rightarrow T^*(M)$  is for each  $u \in M$  injective, it has the unique inverse mapping  $(p^*(u))^{-1}$  upon its image  $p^*(u)T_{p(u)}^*(N) \subset T_u^*(M)$ . Thereby for each  $u \in M$  one can define a morphism  $p_{\mathcal{A}} : (T^*(M), \varphi_g^*) \rightarrow T^*(N)$  as

$$(6.27) \quad p_{\mathcal{A}}(u) : \alpha^{(1)}(u) \rightarrow (p^*(u))^{-1}[1 - \mathcal{A}^*(u)\hat{u}^*(e)]\alpha^{(1)}(u).$$

Based on the definition (6.27) one can easily check that the diagram

$$(6.28) \quad \begin{array}{ccc} T^*(M) & \xrightarrow{p_{\mathcal{A}}} & T^*(N) \\ pr_M \downarrow & & \downarrow pr_N \\ M & \xrightarrow{p} & N \end{array}$$

is commutative.

Let an element  $\xi \in \mathcal{G}^*$  be  $G$ -invariant, that is  $Ad_{g^{-1}}^* \xi = \xi$  for all  $g \in G$ . Denote also by  $p_{\mathcal{A}}^{\xi}$  the restriction of the mapping (6.27) upon the subset  $\mathcal{M}_{\xi} := l^{-1}(\xi) \in T^*(M)$ , that is  $p_{\mathcal{A}}^{\xi} : \mathcal{M}_{\xi} \rightarrow T^*(N)$ , where for all  $u \in M$

$$(6.29) \quad p_{\mathcal{A}}^{\xi}(u) : l^{-1}(\xi) \rightarrow (p^*(u))^{-1}[1 - \mathcal{A}^*(u)\hat{u}^*(e)]l^{-1}(\xi).$$

Now one can characterize the structure of the reduced phase space  $\bar{\mathcal{M}}_{\xi} := l^{-1}(\xi)/G$  by means of the following lemma.

**Lemma 6.4.** *The mapping  $p_{\mathcal{A}}^{\xi}(u) : \mathcal{M}_{\xi} \rightarrow T^*(N)$ , where  $\mathcal{M}_{\xi} := l^{-1}(\xi)$ , is a principal fiber  $G$ -bundle with the reduced space  $\bar{\mathcal{M}}_{\xi}$ , being diffeomorphic to  $T^*(N)$ .*

Denote by  $\langle \cdot, \cdot \rangle_{\mathcal{G}}$  the standard  $Ad$ -invariant non-degenerate scalar product on  $\mathcal{G} \times \mathcal{G}$ . Based on Lemma 6.4 one derives the following characteristic theorem.

**Theorem 6.5.** *Given a principal fiber  $G$ -bundle with a connection  $\Gamma(\mathcal{A})$  and a  $G$ -invariant element  $\xi \in \mathcal{G}^*$ , then every such connection  $\Gamma(\mathcal{A})$  defines a symplectomorphism  $\nu_{\xi} : \bar{\mathcal{M}}_{\xi} \rightarrow T^*(N)$  between the reduced phase space  $\bar{\mathcal{M}}_{\xi}$  and cotangent bundle  $T^*(N)$ , where  $l : T^*(M) \rightarrow \mathcal{G}^*$  is the naturally associated momentum mapping for the group  $G$ -action on  $M$ . Moreover, the following equality*

$$(6.30) \quad (p_{\mathcal{A}}^{\xi})(d \, pr^* \beta^{(1)} + pr^* \Omega_{\xi}^{(2)}) = d \, pr^* \alpha^{(1)} \Big|_{l^{-1}(\xi)}$$

holds for the canonical 1-forms  $\beta^{(1)} \in \Lambda^1(N)$  and  $\alpha^{(1)} \in \Lambda^1(M)$ , where  $\Omega_{\xi}^{(2)} := \langle \xi, \Omega^{(2)} \rangle_{\mathcal{G}}$  is the  $\xi$ -component of the corresponding curvature form  $\Omega^{(2)} \in \Lambda^2(N) \otimes \mathcal{G}$ .

*Proof.* One has that on  $l^{-1}(\xi) \subset M$  the following expression, due to (6.27), holds:

$$p^*(u)p_{\mathcal{A}}^{\xi}(\alpha^{(1)}(u)) = p^*(u)\beta^{(1)}(pr_N(u)) = \alpha^{(1)}(u) - \mathcal{A}^*(u)\hat{u}^*(e)\alpha^{(1)}(u)$$

for any  $\beta^{(1)} \in T^*(N)$  and all  $u \in M_{\xi} := p_M l^{-1}(\xi) \subset M$ . Thus we easily get that

$$\alpha^{(1)}(u) = (p_{\mathcal{A}}^{\xi})^{-1}\beta^{(1)}(p_N(u)) = p^*(u)\beta^{(1)}(pr_N(u)) + \langle \xi, \mathcal{A}(u) \rangle_{\mathcal{G}}$$

for all  $u \in M_{\xi}$ . Recall now that in virtue of (6.28) on the manifold  $M_{\xi}$  there hold relationships:

$$p \circ pr_{M_{\xi}} = pr_N \circ p_{\mathcal{A}}^{\xi}, \quad pr_{M_{\xi}}^* \circ p^* = (p_{\mathcal{A}}^{\xi})^* \circ pr_N^*.$$

Therefore we can now write down that

$$\begin{aligned} pr_{M_{\xi}}^* \alpha^{(1)}(u) &= pr_{M_{\xi}}^* \beta^{(1)}(p_N(u)) + pr_{M_{\xi}}^* \langle \xi, \mathcal{A}(u) \rangle_{\mathcal{G}} \\ &= (p_{\mathcal{A}}^{\xi})^* (pr_N^* \beta^{(1)})(u) + pr_{M_{\xi}}^* \langle \xi, \mathcal{A}(u) \rangle_{\mathcal{G}}, \end{aligned}$$

whence taking the external differential, one arrives at the following equalities:

$$\begin{aligned} d pr_{M_{\xi}}^* \alpha^{(1)}(u) &= (p_{\mathcal{A}}^{\xi})^* d(pr_N^* \beta^{(1)})(u) + pr_{M_{\xi}}^* \langle \xi, d\mathcal{A}(u) \rangle_{\mathcal{G}} \\ &= (p_{\mathcal{A}}^{\xi})^* d(pr_N^* \beta^{(1)})(u) + pr_{M_{\xi}}^* \langle \xi, \Omega(p(u)) \rangle_{\mathcal{G}} \\ &= (p_{\mathcal{A}}^{\xi})^* d(pr_N^* \beta^{(1)})(u) + pr_{M_{\xi}}^* p^* \langle \xi, \Omega, \rangle_{\mathcal{G}}(u) \\ &= (p_{\mathcal{A}}^{\xi})^* d(pr_N^* \beta^{(1)})(u) + (p_{\mathcal{A}}^{\xi})^* pr_N^* \langle \xi, \Omega \rangle_{\mathcal{G}}(u) \\ &= (p_{\mathcal{A}}^{\xi})^* [d(pr_N^* \beta^{(1)})(u) + pr_N^* \langle \xi, \Omega \rangle_{\mathcal{G}}(u)]. \end{aligned}$$

When deriving the above expression we made use of the following property satisfied by the curvature 2-form  $\Omega \in \Lambda^2(M) \otimes \mathcal{G}$ :

$$\begin{aligned} \langle \xi, d\mathcal{A}(u) \rangle_{\mathcal{G}} &= \langle \xi, d\mathcal{A}(u) + \mathcal{A}(u) \wedge \mathcal{A}(u) \rangle_{\mathcal{G}} - \langle \xi, \mathcal{A}(u) \wedge \mathcal{A}(u) \rangle_{\mathcal{G}} \\ &= \langle \xi, \Omega(p_N(u)) \rangle_{\mathcal{G}} = \langle \xi, pr_N^* \Omega \rangle_{\mathcal{G}}(u) \end{aligned}$$

at any  $u \in M_{\xi}$ , since for any  $A, B \in \mathcal{G}$  there holds  $\langle \xi, [A, B] \rangle_{\mathcal{G}} = \langle Ad^* A \xi, B \rangle_{\mathcal{G}} = 0$  in virtue of the invariance condition  $Ad_G^* \xi = \xi$ . Thereby the proof is finished.  $\square$

*Remark 6.6.* As the canonical 2-form  $d pr^* \alpha^{(1)} \in \Lambda^2(T^*(M))$  is  $G$ -invariant on  $T^*(M)$  due to construction, it is evident that its restriction upon the  $G$ -invariant submanifold  $\mathcal{M}_{\xi} \subset T^*(M)$  will be effectively defined only on the reduced space  $\bar{\mathcal{M}}_{\xi}$ , that ensures the validity of the equality sign in (6.30).

As a consequence of Theorem 6.5 one can formulate the following useful for applications theorems.

**Theorem 6.7.** *Let an element  $\xi \in \mathcal{G}^*$  have the isotropy group  $G_{\xi}$  acting on the subset  $\mathcal{M}_{\xi} \subset T^*(M)$  freely and properly, so that the reduced phase space  $(\bar{\mathcal{M}}_{\xi}, \sigma_{\xi}^{(2)})$  where, by definition,  $\bar{\mathcal{M}}_{\xi} := l^{-1}(\xi)/G_{\xi}$ , is symplectic whose symplectic structure is defined as*

$$(6.31) \quad \sigma_{\xi}^{(2)} := d pr^* \alpha^{(1)} \Big|_{l^{-1}(\xi)}.$$

*If a principal fiber bundle  $p : (M, \varphi) \rightarrow N$  has a structure group coinciding with  $G_{\xi}$ , then the reduced symplectic space  $(\bar{\mathcal{M}}_{\xi}, \sigma_{\xi}^{(2)})$  is symplectomorphic to the cotangent symplectic space  $(T^*(N), \omega_{\xi}^{(2)})$ , where*

$$(6.32) \quad \omega_{\xi}^{(2)} = d pr^* \beta^{(1)} + pr^* \Omega_{\xi}^{(2)},$$

*and the corresponding symplectomorphism is given by a relation like (6.30).*

**Theorem 6.8.** *In order that two symplectic spaces  $(\bar{\mathcal{M}}_{\xi}, \sigma_{\xi}^{(2)})$  and  $(T^*(N), dpr^* \beta^{(1)})$  were symplectomorphic, it is necessary and sufficient that the element  $\xi \in \ker h$ , where for  $G$ -invariant element  $\xi \in \mathcal{G}^*$  the mapping  $h : \xi \rightarrow [\Omega_{\xi}^{(2)}] \in H^2(N; \mathbb{Z})$ , with  $H^2(N; \mathbb{Z})$  being the cohomology class of 2-forms on the manifold  $N$ .*

**6.3. The Hamiltonian analysis of the Maxwell electromagnetic dynamical systems.** We take the Maxwell electromagnetic equations to be

$$(6.33) \quad \begin{aligned} \partial E / \partial t &= \nabla \times B - j, & \partial B / \partial t &= -\nabla \times E, \\ < \nabla, E > &= \rho, & < \nabla, B > &= 0, \end{aligned}$$

on the cotangent phase space  $T^*(N)$  to  $N \subset T(D; \mathbb{E}^3)$ , being the smooth manifold of smooth vector fields on an open domain  $D \subset \mathbb{R}^3$ , all expressed in the light speed units. Here  $(E, B) \in T^*(N)$  is a vector of electric and magnetic fields,  $\rho : D \rightarrow \mathbb{R}$  and  $j : D \rightarrow \mathbb{E}^3$  are, simultaneously, fixed charge and current densities in the domain  $D$ , satisfying the equation of continuity

$$(6.34) \quad \partial \rho / \partial t + < \nabla, j > = 0,$$

holding for all  $t \in \mathbb{R}$ , where we denoted by the sign " $\nabla$ " the gradient operation with respect to a variable  $x \in D$ , by the sign " $\times$ " the usual vector product in  $\mathbb{E}^3 := (\mathbb{R}^3, < \cdot, \cdot >)$ , being the standard three-dimensional Euclidean vector space  $\mathbb{R}^3$  endowed with the usual scalar product  $< \cdot, \cdot >$ .

Aiming to represent equations (6.33) as those on reduced symplectic space, we define an appropriate configuration space  $M \subset \mathcal{T}(D; \mathbb{E}^3)$  with a vector potential field coordinate  $A \in M$ . The cotangent space  $T^*(M)$  may be identified with pairs  $(A; Y) \in T^*(M)$ , where  $Y \in \mathcal{T}^*(D; \mathbb{E}^3)$  is a suitable vector field density in  $D$ . On the space  $T^*(M)$  there exists the canonical symplectic form  $\omega^{(2)} \in \Lambda^2(T^*(M))$ , allowing, owing to the definition of the Liouville from

$$(6.35) \quad \lambda(\alpha^{(1)})(A; Y) = \int_D d^3x (< Y, dA > := (Y, dA),$$

the canonical expression

$$(6.36) \quad \omega^{(2)} := d\lambda(\alpha^{(1)}) = (dY, \wedge dA).$$

Here we denoted by " $\wedge$ " the usual external differentiation, by  $d^3x$ ,  $x \in D$ , the Lebesgue measure in the domain  $D$  and by  $pr : T^*(M) \rightarrow M$  the standard projection upon the base space  $M$ . Define now a Hamiltonian function  $\tilde{H} \in \mathcal{D}(T^*(M))$  as

$$(6.37) \quad H(A, Y) = 1/2[(Y, Y) + (\nabla \times A, \nabla \times A) + (< \nabla, A >, < \nabla, A >)],$$

describing the well-known Maxwell equations in vacuum, if the densities  $\rho = 0$  and  $j = 0$ . Really, owing to (6.36) one easily obtains from (6.37) that

$$(6.38) \quad \begin{aligned} \partial A / \partial t &: = \delta H / \delta Y = Y, \\ \partial Y / \partial t &: = -\delta H / \delta A = -\nabla \times B + \nabla < \nabla, A >, \end{aligned}$$

being true wave equations in vacuum, where we put, by definition,

$$(6.39) \quad B := \nabla \times A,$$

being the corresponding magnetic field. Now defining

$$(6.40) \quad E := -Y - \nabla W$$

for some function  $W : M \rightarrow \mathbb{R}$  as the corresponding electric field, the system of equations (6.38) will become, owing to definition (6.39),

$$(6.41) \quad \partial B / \partial t = -\nabla \times E, \quad \partial E / \partial t = \nabla \times B,$$

exactly coinciding with the Maxwell equations in vacuum, if the Lorenz condition

$$(6.42) \quad \partial W / \partial t + < \nabla, A > = 0$$

is involved.

Since definition (6.40) was essentially imposed rather than arising naturally from the Hamiltonian approach and our equations are valid only for a vacuum, we shall try to improve upon these matters by employing the reduction approach devised above. Namely, we start with the Hamiltonian (6.37) and observe that it is invariant with respect to the following abelian symmetry group  $G := \exp \mathcal{G}$ , where  $\mathcal{G} \simeq C^{(1)}(D; \mathbb{R})$ , acting on the base manifold  $M$  naturally lifted to  $T^*(M)$ : for any  $\psi \in \mathcal{G}$  and  $(A, Y) \in T^*(M)$

$$(6.43) \quad \varphi_\psi(A) := A + \nabla \psi, \quad \varphi_\psi(Y) = Y.$$



The 1-form (6.35) under transformation (6.43) also is invariant since

$$(6.44) \quad \begin{aligned} \varphi_\psi^* \lambda(\alpha^{(1)})(A, Y) &= (Y, dA + \nabla d\psi) = \\ &= (Y, dA) - \langle \nabla, Y \rangle, d\psi = \lambda(\alpha^{(1)})(A, Y), \end{aligned}$$

where we made use of the condition  $d\psi \simeq 0$  in  $\Lambda^1(T^*(M))$  for any  $\psi \in \mathcal{G}$ . Thus, the corresponding momentum mapping (6.22) is given as

$$(6.45) \quad l(A, Y) = - \langle \nabla, Y \rangle$$

for all  $(A, Y) \in T^*(M)$ . If  $\rho \in \mathcal{G}^*$  is fixed, one can define the reduced phase space  $\bar{\mathcal{M}}_\rho := l^{-1}(\rho)/G$ , since evidently, the isotropy group  $G_\rho = G$ , owing to its commutativity and the condition (6.43). Consider now a principal fiber bundle  $p : M \rightarrow N$  with the abelian structure group  $G$  and a base manifold  $N$  taken as

$$(6.46) \quad N := \{B \in \mathcal{T}(D; \mathbb{E}^3) : \langle \nabla, B \rangle = 0, \langle \nabla, E(S) \rangle = \rho\},$$

where, by definition,

$$(6.47) \quad p(A) = B = \nabla \times A.$$

We can construct a connection 1-form  $\mathcal{A} \in \Lambda^1(M) \otimes \mathcal{G}$  on this bundle, where for all  $A \in M$

$$(6.48) \quad \mathcal{A}(A) \cdot \hat{A}_*(l) = 1, \quad d \langle \mathcal{A}(A), \rho \rangle = \Omega_\rho^{(2)}(A) \in H^2(M; \mathbb{Z}),$$

where  $\mathcal{A}(A) \in \Lambda^1(M)$  is some differential 1-form, which we choose in the following form:

$$(6.49) \quad \mathcal{A}(A) := -(W, d \langle \nabla, A \rangle),$$

where  $W \in C^{(1)}(D; \mathbb{R})$  is some scalar function, still not defined. As a result, the Liouville form (6.35) transforms into

$$(6.50) \quad \lambda(\tilde{\alpha}_\rho^{(1)}) := (Y, dA) - (W, d \langle \nabla, A \rangle) = (Y + \nabla W, dA) := (\tilde{Y}, dA), \quad \tilde{Y} := Y + \nabla W,$$

giving rise to the corresponding canonical symplectic structure on  $T^*(M)$  as

$$(6.51) \quad \tilde{\omega}_\rho^{(2)} := d\lambda(\tilde{\alpha}_\rho^{(1)}) = (d\tilde{Y}, \wedge dA).$$

Respectively, the Hamiltonian function (6.37), as a function on  $T^*(M)$ , transforms into

$$(6.52) \quad \tilde{H}_\rho(A, \tilde{Y}) = 1/2[(\tilde{Y}, \tilde{Y}) + (\nabla \times A, \nabla \times A) + \langle \nabla, A \rangle, \langle \nabla, A \rangle],$$

coinciding with the well-known Dirac-Fock-Podolsky [28, 43] Hamiltonian expression. The corresponding Hamiltonian equations on the cotangent space  $T^*(M)$

$$\begin{aligned} \partial A / \partial t &: = \delta \tilde{H} / \delta \tilde{Y} = \tilde{Y}, \quad \tilde{Y} := -E - \nabla W, \\ \partial \tilde{Y} / \partial t &: = -\delta \tilde{H} / \delta A = -\nabla \times (\nabla \times A) + \nabla \langle \nabla, A \rangle, \end{aligned}$$

describe true wave processes related to the Maxwell equations in vacuum, which do not take into account boundary charge and current densities conditions. Really, from (6.52) we obtain that

$$(6.53) \quad \partial^2 A / \partial t^2 - \nabla^2 A = 0 \implies \partial E / \partial t + \nabla(\partial W / \partial t + \langle \nabla, A \rangle) = -\nabla \times B,$$

giving rise to the true vector potential wave equation, but the electromagnetic Faraday induction law is satisfied if one to impose additionally the Lorenz condition (6.42).

To remedy this situation, we will apply to this symplectic space the reduction technique devised in Section 1. Namely, owing to Theorem 6.7, the constructed above cotangent manifold  $T^*(N)$  is symplectomorphic to the corresponding reduced phase space  $\bar{\mathcal{M}}_\rho$ , that is

$$(6.54) \quad \bar{\mathcal{M}}_\rho \simeq \{(B; S) \in T^*(N) : \langle \nabla, E(S) \rangle = \rho, \langle \nabla, B \rangle = 0\}$$

with the reduced canonical symplectic 2-form

$$(6.55) \quad \omega_\rho^{(2)}(B, S) = (dB, \wedge dS) = d\lambda(\alpha_\rho^{(1)})(B, S), \quad \lambda(\alpha_\rho^{(1)})(B, S) := -(S, dB),$$

where we put, by definition,

$$(6.56) \quad \nabla \times S + F + \nabla W = -\tilde{Y} := E + \nabla W, \quad \langle \nabla, F \rangle = \rho,$$

for some fixed vector mapping  $F \in C^{(1)}(D; \mathbb{E}^3)$ , depending on the imposed boundary conditions. The result (6.55) follows right away upon substituting the expression for the electric field  $E =$

$\nabla \times S + F$  into the symplectic structure (6.51), and taking into account that  $dF = 0$  in  $\Lambda^1(M)$ . The Hamiltonian function (6.52) reduces, respectively, to the following symbolic form:

$$(6.57) \quad \begin{aligned} H_\rho(B, S) &= 1/2[(B, B) + (\nabla \times S + F + \nabla W, \nabla \times S + F + \nabla W) + \\ &+ (\langle \nabla, (\nabla \times)^{-1} B \rangle, \langle \nabla, (\nabla \times)^{-1} B \rangle)], \end{aligned}$$

where " $(\nabla \times)^{-1}$ " means, by definition, the corresponding inverse curl-operation, mapping [104] the divergence-free subspace  $C_{div}^{(1)}(D; \mathbb{E}^3) \subset C^{(1)}(D; \mathbb{E}^3)$  into itself. As a result from (6.57), the Maxwell equations (6.33) become a canonical Hamiltonian system upon the reduced phase space  $T^*(N)$ , endowed with the canonical symplectic structure (6.55) and the modified Hamiltonian function (6.57). Really, one easily obtains that

$$(6.58) \quad \begin{aligned} \partial S / \partial t &: = \delta H / \delta B = B - (\nabla \times)^{-1} \nabla \langle \nabla, (\nabla \times)^{-1} B \rangle, \\ \partial B / \partial t &: = -\delta H / \delta S = -\nabla \times (\nabla \times S + F + \nabla W) := -\nabla \times E, \end{aligned}$$

where we make use of the definition  $E = \nabla \times S + F$  and the elementary identity  $\nabla \times \nabla = 0$ . Thus, the second equation of (6.58) coincides with the second Maxwell equation of (6.33) in the classical form

$$\partial B / \partial t = -\nabla \times E.$$

Moreover, from (6.56), owing to (6.58), one obtains via the differentiation with respect to  $t \in \mathbb{R}$  that

$$(6.59) \quad \begin{aligned} \partial E / \partial t &= \partial F / \partial t + \nabla \times \partial S / \partial t = \\ &= \partial F / \partial t + \nabla \times B, \end{aligned}$$

as well as, owing to (6.34),

$$(6.60) \quad \langle \nabla, \partial F / \partial t \rangle = \partial \rho / \partial t = -\langle \nabla, j \rangle.$$

So, we can find from (6.60) that, up to non-essential curl-terms  $\nabla \times (\cdot)$ , the following relationship

$$(6.61) \quad \partial F / \partial t = -j$$

holds. Really, the current density vector  $j \in C^{(1)}(D; \mathbb{E}^3)$ , owing to the equation of continuity (6.34), is defined up to curl-terms  $\nabla \times (\cdot)$  which can be included into the right-hand side of (6.61). Having now substituted (6.61) into (6.59), we obtain exactly the first Maxwell equation of (6.33):

$$(6.62) \quad \partial E / \partial t = \nabla \times B - j,$$

being supplemented, naturally, with the external boundary constraint conditions

$$(6.63) \quad \begin{aligned} \langle \nabla, B \rangle &= 0, \quad \langle \nabla, E \rangle = \rho, \\ \partial \rho / \partial t + \langle \nabla, j \rangle &= 0, \end{aligned}$$

owing to the continuity relationship (6.34) and definition (6.54).

Concerning the wave equations, related to the Hamiltonian system (6.58), we obtain the following: the electric field  $E$  is recovered from the second equation as

$$(6.64) \quad E := -\partial A / \partial t - \nabla W,$$

where  $W \in C^{(1)}(D; \mathbb{R})$  is some smooth function, depending on the vector field  $A \in M$ . To retrieve this dependence, we substitute (6.61) into equation (6.62), having taken into account that  $B = \nabla \times A$ :

$$(6.65) \quad \partial^2 A / \partial t^2 - \nabla(\partial W / \partial t + \langle \nabla, A \rangle) = \nabla^2 A + j.$$

With the above, if we now impose the Lorenz condition (6.42), we obtain from (6.65) the corresponding true wave equations in the space-time, taking into account the external charge and current density conditions (6.63).

Notwithstanding our progress so far, the problem of fulfilling the Lorenz constraint (6.42) naturally within the canonical Hamiltonian formalism still remains to be completely solved. To this end, we are compelled to analyze the structure of the Liouville 1-form (6.50) for Maxwell equations in vacuum on a slightly extended functional manifold  $M \times L$ . As a first step, we rewrite 1-form (6.50) as

$$(6.66) \quad \begin{aligned} \lambda(\tilde{\alpha}_\rho^{(1)}) &: = (\tilde{Y}, dA) = (Y + \nabla W, dA) = (Y, dA) + \\ &+ (W, -d \langle \nabla, A \rangle) := (Y, dA) + (W, d\chi), \end{aligned}$$

where we put, by definition,

$$(6.67) \quad \chi := - \langle \nabla, A \rangle.$$

Considering now the elements  $(Y, A; \chi, W) \in T^*(M \times L)$  as new canonical variables on the extended cotangent phase space  $T^*(M \times L)$ , where  $L := C^{(1)}(D; \mathbb{R})$ , we can rewrite the symplectic structure (6.51) in the following canonical form

$$(6.68) \quad \tilde{\omega}_\rho^{(2)} := d\lambda(\tilde{\alpha}_\rho^{(1)}) = (dY, \wedge dA) + (dW, \wedge d\chi).$$

Subject to the Hamiltonian function (6.52) we obtain the expression

$$(6.69) \quad H(A, Y; \chi, W) = 1/2[(Y - \nabla W, Y - \nabla W) + (\nabla \times A, \nabla \times A) + (\chi, \chi)],$$

with respect to which the corresponding Hamiltonian equations take the form:

$$(6.70) \quad \begin{aligned} \partial A / \partial t &: = \delta H / \delta Y = Y - \nabla W, & Y &:= -E, \\ \partial Y / \partial t &: = -\delta H / \delta A = -\nabla \times (\nabla \times A), \\ \partial \chi / \partial t &: = \delta H / \delta W = \langle \nabla, Y - \nabla W \rangle, \\ \partial W / \partial t &: = -\delta H / \delta \chi = -\chi. \end{aligned}$$

From (6.70) we obtain, owing to external boundary conditions (6.63), successively that

$$(6.71) \quad \begin{aligned} \partial B / \partial t + \nabla \times E &= 0, & \partial^2 W / \partial t^2 - \nabla^2 W &= \rho, \\ \partial E / \partial t - \nabla \times B &= 0, & \partial^2 A / \partial t^2 - \nabla^2 A &= -\nabla(\partial W / \partial t + \langle \nabla, A \rangle). \end{aligned}$$

As is seen, these equations describe electromagnetic Maxwell equations in vacuum, but without the Lorenz condition (6.42). Thereby, as above, we will apply to the symplectic structure (6.68) the reduction technique devised in Section 1. We obtain that under transformations (6.56) the corresponding reduced manifold  $\bar{\mathcal{M}}_\rho$  becomes endowed with the symplectic structure

$$(6.72) \quad \bar{\omega}_\rho^{(2)} := (dB, \wedge dS) + (dW, \wedge d\chi),$$

and the Hamiltonian (6.69) assumes the form

$$(6.73) \quad H(S, B; \chi, W) = 1/2[(\nabla \times S + F + \nabla W, \nabla \times S + F + \nabla W) + (B, B) + (\chi, \chi)],$$

whose Hamiltonian equations

$$(6.74) \quad \begin{aligned} \partial S / \partial t &: = \delta H / \delta B = B, & \partial W / \partial t &:= -\delta H / \delta \chi = -\chi, \\ \partial B / \partial t &: = -\delta H / \delta S = -\nabla \times (\nabla \times S + F + \nabla W) = -\nabla \times E, \\ \partial \chi / \partial t &: = \delta H / \delta W = -\langle \nabla, \nabla \times S + F + \nabla W \rangle = -\langle \nabla, E \rangle - \Delta W, \end{aligned}$$

coincide completely with Maxwell equations (6.33) under conditions (6.56), describing true wave processes in vacuum, as well as the electromagnetic Maxwell equations, taking into account *a priori* both the imposed external boundary conditions (6.63) and the Lorenz condition (6.42), solving the problem mentioned in [28, 43]. Really, it is easy to obtain from (6.74) that

$$(6.75) \quad \begin{aligned} \partial^2 W / \partial t^2 - \Delta W &= \rho, & \partial W / \partial t + \langle \nabla, A \rangle &= 0, \\ \nabla \times B &= j + \partial E / \partial t, & \partial B / \partial t &= -\nabla \times E, \end{aligned}$$

Based now on (6.75) and (6.63) one can easily calculate [24, 125] the magnetic wave equation

$$(6.76) \quad \partial^2 A / \partial t^2 - \Delta A = j,$$

supplementing the suitable wave equation on the scalar potential  $W \in L$ , finishing the calculations. Thus, we can formulate the following proposition.

**Proposition 6.9.** *The electromagnetic Maxwell equations (6.33) jointly with Lorenz condition (6.42) are equivalent to the Hamiltonian system (6.74) with respect to the canonical symplectic structure (6.72) and Hamiltonian function (6.73), which correspondingly reduce to electromagnetic equations (6.75) and (6.76) under external boundary conditions (6.63).*

The obtained above result can be, eventually, used for developing an alternative quantization procedure of Maxwell electromagnetic equations, being free of some quantum operator problems, discussed in detail in [28]. We hope to consider this aspect of quantization problem in a specially devoted study.

*Remark 6.10.* If one considers a motion of a charged point particle under a Maxwell field, it is convenient to introduce a trivial fiber bundle structure  $\pi : M \rightarrow N$ , such that  $M \simeq N \times G$ ,  $N := D \subset \mathbb{R}^3$ , with  $G := \mathbb{R} \setminus \{0\}$  being the corresponding (abelian) structure Lie group. An analysis similar to the above gives rise to the reduced (on the reduced space  $\bar{\mathcal{M}}_\xi := l^{-1}(\xi)/G \simeq T^*(N)$ ,  $\xi \in \mathcal{G}$ ) symplectic structure

$$\omega^{(2)}(q, p) = \langle dp, \wedge dq \rangle + d \langle \xi, \mathcal{A}(q, g) \rangle_{\mathcal{G}},$$

where  $\mathcal{A}(q, g) := \langle A(q), dq \rangle + g^{-1}dg$  is a suitable connection 1-form on phase space  $M$ , with  $(q, p) \in T^*(N)$  and  $g \in G$ . The corresponding canonical Poisson brackets on  $T^*(N)$  are easily found to be

$$(6.77) \quad \{q^i, q^j\} = 0, \quad \{p_j, q^i\} = \delta_j^i, \quad \{p_i, p_j\} = F_{ji}(q)$$

for all  $(q, p) \in T^*(N)$ . If one introduces a new momentum variable  $\tilde{p} := p + A(q)$  on  $T^*(N) \ni (q, p)$ , it is easy to verify that  $\omega_\xi^{(2)} \rightarrow \tilde{\omega}_\xi^{(2)} := \langle d\tilde{p}, \wedge dq \rangle$ , giving rise to the following Poisson brackets [96, 132, 131]:

$$(6.78) \quad \{q^i, q^j\} = 0, \quad \{\tilde{p}_j, q^i\} = \delta_j^i, \quad \{\tilde{p}_i, \tilde{p}_j\} = 0,$$

where  $i, j = \overline{1, 3}$ , iff the standard Maxwell field equations

$$(6.79) \quad \partial F_{ij}/\partial q_k + \partial F_{jk}/\partial q_i + \partial F_{ki}/\partial q_j = 0$$

are satisfied on  $N$  for all  $i, j, k = \overline{1, 3}$  with the curvature tensor  $F_{ij}(q) := \partial A_j/\partial q^i - \partial A_i/\partial q^j$ ,  $i, j = \overline{1, 3}$ ,  $q \in N$ .

Such a construction permits a natural generalization to the case of non-abelian structure Lie group yielding a description of Yang-Mills field equations within the reduction approach, to which we proceed below.

**6.4. The Hamiltonian analysis of the Yang-Mills type dynamical systems.** As above, we start with defining a phase space  $M$  of a particle under a Yang-Mills field in a region  $D \subset \mathbb{R}^3$  as  $M := D \times G$ , where  $G$  is a (not in general semi-simple) Lie group, acting on  $M$  from the right. Over the space  $M$  one can define quite naturally a connection  $\Gamma(\mathcal{A})$  if we consider the following trivial principal fiber bundle  $p : M \rightarrow N$ , where  $N := D$ , with the structure group  $G$ . Namely, if  $g \in G$ ,  $q \in N$ , then a connection 1-form on  $M \ni (q, g)$  can be expressed [62, 119, 20, 110] as

$$(6.80) \quad \mathcal{A}(q; g) := g^{-1} \left( d + \sum_{i=1}^n a_i A^{(i)}(q) \right) g,$$

where  $\{a_i \in \mathcal{G} : i = \overline{1, n}\}$  is a basis of the Lie algebra  $\mathcal{G}$  of the Lie group  $G$ , and  $A_i : D \rightarrow \Lambda^1(D)$ ,  $i = \overline{1, n}$ , are the Yang-Mills fields on the physical space  $D \subset \mathbb{R}^3$ .

Now one defines the natural left invariant Liouville form on  $M$  as

$$(6.81) \quad \alpha^{(1)}(q; g) := \langle p, dq \rangle + \langle y, g^{-1}dg \rangle_{\mathcal{G}},$$

where  $y \in T^*(G)$  and  $\langle \cdot, \cdot \rangle_{\mathcal{G}}$  denotes, as before, the usual Ad-invariant non-degenerate bilinear form on  $\mathcal{G}^* \times \mathcal{G}$ , as, evidently,  $g^{-1}dg \in \Lambda^1(G) \otimes \mathcal{G}$ . The main assumption we need to proceed is that the connection 1-form is compatible with the Lie group  $G$  action on  $M$ . The latter means that the condition

$$(6.82) \quad R_h^* \mathcal{A}(q; g) = Ad_{h^{-1}} \mathcal{A}(q; g)$$

is satisfied for all  $(q, g) \in M$  and  $h \in G$ , where  $R_h : G \rightarrow G$  means the right translation by an element  $h \in G$  on the Lie group  $G$ .

Having stated all preliminary conditions needed for the reduction Theorem 6.7 to be applied to our model, suppose that the Lie group  $G$  canonical action on  $M$  is naturally lifted to that on the cotangent space  $T^*(M)$  endowed due to (endowed owing to (6.35) with the following  $G$ -invariant canonical symplectic structure:

$$(6.83) \quad \begin{aligned} \omega^{(2)}(q, p; g, y) &: = d pr^* \alpha^{(1)}(q, p; g, y) = \langle dp, \wedge dq \rangle + \\ &+ \langle dy, \wedge g^{-1}dg \rangle_{\mathcal{G}} + \langle y dg^{-1}, \wedge dg \rangle_{\mathcal{G}} \end{aligned}$$

for all  $(q, p; g, y) \in T^*(M)$ . Take now an element  $\xi \in \mathcal{G}^*$  and assume that its isotropy subgroup  $G_\xi = G$ , that is  $Ad_h^* \xi = \xi$  for all  $h \in G$ . In the general case such an element  $\xi \in \mathcal{G}^*$  cannot exist but trivial  $\xi = 0$ , as it happens, for instance, in the case of the Lie group  $G = SL_2(\mathbb{R})$ . Then one

can construct the reduced phase space  $l^{-1}(\xi)/G$  symplectomorphic to  $(T^*(N), \omega_\xi^{(2)})$ , where owing to (6.30) for any  $(q, p) \in T^*(N)$

$$(6.84) \quad \begin{aligned} \omega_\xi^{(2)}(q, p) &= \langle dp, \wedge dq \rangle + \langle \Omega^{(2)}(q), \xi \rangle_{\mathcal{G}} = \\ &= \langle dp, \wedge dq \rangle + \sum_{s=1}^n \sum_{i,j=1}^3 e_s F_{ij}^{(s)}(q) dq^i \wedge dq^j. \end{aligned}$$

In the above we have expanded the element  $\xi = \sum_{i=1}^n e_i a^i \in \mathcal{G}^*$  with respect to the bi-orthogonal basis  $\{a^i \in \mathcal{G}^*, a_j \in \mathcal{G} : \langle a^i, a_j \rangle_{\mathcal{G}} = \delta_j^i, i, j = \overline{1, n}\}$ , with  $e_i \in \mathbb{R}, i = \overline{1, 3}$ , being some constants, and we, as well, denoted by  $F_{ij}^{(s)}(q), i, j = \overline{1, 3}, s = \overline{1, n}$ , the corresponding curvature 2-form  $\Omega^{(2)} \in \Lambda^2(N) \otimes \mathcal{G}$  components, that is

$$(6.85) \quad \Omega^{(2)}(q) := \sum_{s=1}^n \sum_{i,j=1}^3 a_s F_{ij}^{(s)}(q) dq^i \wedge dq^j$$

for any point  $q \in N$ . Summarizing the calculations accomplished above, we can formulate the following result.

**Theorem 6.11.** *Suppose the Yang-Mills field (6.80) on the fiber bundle  $p : M \rightarrow N$  with  $M = D \times G$  is invariant with respect to the Lie group  $G$  action  $G \times M \rightarrow M$ . Suppose also that an element  $\xi \in \mathcal{G}^*$  is chosen so that  $Ad_G^* \xi = \xi$ . Then for the naturally constructed momentum mapping  $l : T^*(M) \rightarrow \mathcal{G}^*$  (being equivariant) the reduced phase space  $l^{-1}(\xi)/G \simeq T^*(N)$  is endowed with the symplectic structure (6.84), having the following component-wise Poisson brackets form:*

$$(6.86) \quad \{p_i, q^j\}_\xi = \delta_j^i, \quad \{q^i, q^j\}_\xi = 0, \quad \{p_i, p_j\}_\xi = \sum_{s=1}^n e_s F_{ji}^{(s)}(q)$$

for all  $i, j = \overline{1, 3}$  and  $(q, p) \in T^*(N)$ .

The respectively extended Poisson bracket on the whole cotangent space  $T^*(M)$  amounts owing to (6.43) into the following set of Poisson relationships:

$$(6.87) \quad \begin{aligned} \{y_s, y_k\}_\xi &= \sum_{r=1}^n c_{sk}^r y_r, & \{p_i, q^j\}_\xi &= \delta_i^j, \\ \{y_s, p_j\}_\xi &= 0 = \{q^i, q^j\}, & \{p_i, p_j\}_\xi &= \sum_{s=1}^n y_s F_{ji}^{(s)}(q), \end{aligned}$$

where  $i, j = \overline{1, n}, c_{sk}^r \in \mathbb{R}, s, k, r = \overline{1, m}$ , are the structure constants of the Lie algebra  $\mathcal{G}$ , and we made use of the expansion  $A^{(s)}(q) = \sum_{j=1}^n A_j^{(s)}(q) dq^j$  as well we introduced alternative fixed values  $e_i := y_i, i = \overline{1, n}$ . The result (6.87) can be easily seen if one makes a shift within the expression (6.83) as  $\sigma^{(2)} \rightarrow \sigma_{ext}^{(2)}$ , where  $\sigma_{ext}^{(2)} := \sigma^{(2)}|_{\mathcal{A}_0 \rightarrow \mathcal{A}}, \mathcal{A}_0(g) := g^{-1}dg, g \in G$ . Thereby one can obtain in virtue of the invariance properties of the connection  $\Gamma(\mathcal{A})$  that

$$(6.88) \quad \begin{aligned} \sigma_{ext}^{(2)}(q, p; u, y) &= \langle dp, \wedge dq \rangle + d \langle y(g), Ad_{g^{-1}} \mathcal{A}(q; e) \rangle_{\mathcal{G}} = \\ &= \langle dp, \wedge dq \rangle + \langle d Ad_{g^{-1}}^* y(g), \wedge \mathcal{A}(q; e) \rangle_{\mathcal{G}} = \langle dp, \wedge dq \rangle + \sum_{s=1}^m dy_s \wedge du^s + \\ &+ \sum_{j=1}^n \sum_{s=1}^m A_j^{(s)}(q) dy_s \wedge dq^j - \langle Ad_{g^{-1}}^* y(g), \mathcal{A}(q, e) \wedge \mathcal{A}(q, e) \rangle_{\mathcal{G}} + \\ &+ \sum_{k \geq s=1}^m \sum_{l=1}^m y_l c_{sk}^l du^k \wedge du^s + \sum_{s=1}^n \sum_{i \geq j=1}^3 y_s F_{ij}^{(s)}(q) dq^i \wedge dq^j, \end{aligned}$$

where coordinate points  $(q, p; u, y) \in T^*(M)$  are defined as follows:  $\mathcal{A}_0(e) := \sum_{s=1}^m du^s a_s$ ,  $Ad_{g^{-1}}^* y(g) = y(e) := \sum_{s=1}^m y_s a^s$  for any element  $g \in G$ . Hence one gets straightaway the Poisson brackets (2.8) plus additional brackets connected with conjugated sets of variables  $\{u^s \in \mathbb{R} : s = \overline{1, m}\} \in \mathcal{G}^*$  and  $\{y_s \in \mathbb{R} : s = \overline{1, m}\} \in \mathcal{G}$ :

$$(6.89) \quad \{y_s, u^k\}_\xi = \delta_s^k, \quad \{u^k, q^j\}_\xi = 0, \quad \{p_j, u^s\}_\xi = A_j^{(s)}(q), \quad \{u^s, u^k\}_\xi = 0,$$

where  $j = \overline{1, n}$ ,  $k, s = \overline{1, m}$ , and  $q \in N$ .

Note here that the transition suggested above from the symplectic structure  $\sigma^{(2)}$  on  $T^*(N)$  to its extension  $\sigma_{ext}^{(2)}$  on  $T^*(M)$  just consists formally in adding to the symplectic structure  $\sigma^{(2)}$  an exact part, which transforms it into an equivalent one. Looking now at the expressions (6.88), one can infer immediately that an element  $\xi := \sum_{s=1}^m e_s a^s \in \mathcal{G}^*$  will be invariant with respect to the  $Ad^*$ -action of the Lie group  $G$  iff

$$(6.90) \quad \{y_s, y_k\}_\xi|_{y_s=e_s} = \sum_{r=1}^m c_{sk}^r e_r = 0$$

identically for all  $s, k = \overline{1, m}$ ,  $j = \overline{1, n}$  and  $q \in N$ . In this, and only this case, the reduction scheme elaborated above will go through.

Returning our attention to expression (6.89), one can easily write the following exact expression:

$$(6.91) \quad \omega_{ext}^{(2)}(q, p; u, y) = \omega^{(2)}(q, p + \sum_{s=1}^n y_s A^{(s)}(q); u, y),$$

on the phase space  $T^*(M) \ni (q, p; u, y)$ , where we abbreviated  $\langle A^{(s)}(q), dq \rangle$  as  $\sum_{j=1}^n A_j^{(s)}(q) dq^j$ . The transformation like (6.91) was discussed within somewhat different contexts in articles [96, 132] containing also a good background for the infinite dimensional generalization of symplectic structure techniques. Having observed from (6.91) that the simple change of variable

$$(6.92) \quad \tilde{p} := p + \sum_{s=1}^m y_s A^{(s)}(q)$$

of the cotangent space  $T^*(N)$  recasts our symplectic structure (6.88) into the old canonical form (6.83), one obtains that the following new set of canonical Poisson brackets on  $T^*(M) \ni (q, \tilde{p}; u, y)$ :

$$(6.93) \quad \begin{aligned} \{y_s, y_k\}_\xi &= \sum_{r=1}^n c_{sk}^r y_r, & \{\tilde{p}_i, \tilde{p}_j\}_\xi &= 0, & \{\tilde{p}_i, q^j\}_\xi &= \delta_i^j, \\ \{y_s, q^j\}_\xi &= 0 = \{q^i, q^j\}_\xi, & \{u^s, u^k\}_\xi &= 0, & \{y_s, \tilde{p}_j\}_\xi &= 0, \\ \{u^s, q^i\}_\xi &= 0, & \{y_s, u^k\}_\xi &= \delta_s^k, & \{u^s, \tilde{p}_j\}_\xi &= 0, \end{aligned}$$

where  $k, s = \overline{1, m}$  and  $i, j = \overline{1, n}$ , holds iff the non-abelian Yang-Mills type field equations

$$(6.94) \quad \begin{aligned} &\partial F_{ij}^{(s)} / \partial q^l + \partial F_{jl}^{(s)} / \partial q^i + \partial F_{li}^{(s)} / \partial q^j + \\ &+ \sum_{k,r=1}^m c_{kr}^s (F_{ij}^{(k)} A_l^{(r)} + F_{jl}^{(k)} A_i^{(r)} + F_{li}^{(k)} A_j^{(r)}) = 0 \end{aligned}$$

are fulfilled for all  $s = \overline{1, m}$  and  $i, j, l = \overline{1, n}$  on the base manifold  $N$ . This effect of complete reduction of gauge Yang-Mills type variables from the symplectic structure (6.88) is known in literature [96] as the principle of "minimal interaction" and appeared to be useful enough for studying different interacting systems as in [104, 133]. We plan to continue further the study of the geometric properties of reduced symplectic structures connected with such interesting infinite-dimensional coupled dynamical systems of Yang-Mills-Vlasov, Yang-Mills-Bogolubov and Yang-Mills-Josephson types [104, 133] as well as their relationships with associated principal fiber bundles endowed with canonical connection structures.

## 7. THE MAXWELL ELECTROMAGNETIC EQUATIONS AND THE LORENTZ TYPE FORCE DERIVATION - THE FEYNMAN'S APPROACH LEGACY

**7.1. Problem setting.** Still in 1948 R. Feynman presented but not published [49, 50] a very interesting, in some aspect "heretical", quantum-mechanical derivation of the classical Lorentz force acting on a charged particle under influence of an external electromagnetic field. His result was analyzed by many authors [99, 35, 44, 142, 4, 54, 151, 79, 74] from different points of view, including its relativistic generalization [146]. As this problem is completely classical, we reanalyze the Feynman's derivation from the classical Hamiltonian dynamics point of view on the coadjoint

space  $T^*(N)$ ,  $N \subset \mathbb{R}^3$ , and construct its nontrivial generalization compatible with results [125, 19, 121] of Section 1, based on a recently devised vacuum field theory approach [127, 126]. Having further obtained the classical Maxwell electromagnetic equations we supply the complete legacy of the Feynman's approach to the Lorentz force derivation and demonstrate its compatibility with the relativistic generalization, presented in Section 1.

Consider a motion of a charged point particle under a electromagnetic field. For its description, following Section 3, it is convenient to introduce a trivial fiber bundle structure  $\pi: M \rightarrow N$ ,  $M = N \times G$ ,  $N \subset \mathbb{R}^3$ , with the abelian structure group  $G := \mathbb{R} \setminus \{0\}$ , equivariantly acting on the canonically symplectic coadjoint space  $T^*(M)$ , and to endow it with some connection one-form  $A: M \rightarrow T^*(M) \times \mathcal{G}$  as

$$(7.1) \quad \mathcal{A}(q, g) := g^{-1}(d + \alpha^{(1)}(q)) g$$

on the phase space  $M$ , where  $d: \Lambda(M) \rightarrow \Lambda(M)$  is the usual exterior differentiation,  $\alpha^{(1)}: M \rightarrow \Lambda^1(N) \otimes \mathcal{G}$  is some smooth mapping,  $q \in N$  and  $g \in G$ . If  $l: T^*(M) \rightarrow \mathcal{G}^*$  is the related momentum mapping, one can respectively construct the reduced phase space  $\bar{\mathcal{M}}_\xi := l^{-1}(\xi)/G \simeq T^*(N)$ , where  $\xi \in \mathcal{G}^* \simeq \mathbb{R}$  is taken to be fixed, possessing the reduced symplectic structure

$$(7.2) \quad \bar{\omega}_\xi^{(2)}(q, p) = \langle dp, \wedge dq \rangle + d \langle \xi, \alpha^{(1)}(q) \rangle_{\mathcal{G}}.$$

From (7.2) one finds easily the corresponding Poisson brackets on  $T^*(N)$ :

$$(7.3) \quad \{q^i, q^j\}_{\bar{\omega}_\xi^{(2)}} = 0, \quad \{p_j, q^i\}_{\bar{\omega}_\xi^{(2)}} = \delta_j^i, \quad \{p_i, p_j\}_{\bar{\omega}_\xi^{(2)}} = \xi F_{ji}(q)$$

for  $i, j = \overline{1, 3}$  with respect to the reference frame  $\mathcal{K}(t, q)$ , characterized by the phase space coordinates  $(q, p) \in T^*(N)$ . If one introduces a new momentum variable  $\tilde{p} := p + \xi A(q)$  on  $T^*(N) \ni (q, p)$ , where  $\alpha^{(1)}(q) := \langle A(q), dq \rangle \in T_q^*(N)$ , it is easy to verify that  $\bar{\omega}_\xi^{(2)} \rightarrow \tilde{\omega}_\xi^{(2)} := \langle d\tilde{p}, \wedge dq \rangle$ , giving rise to the following "minimal interaction" canonical Poisson brackets [96, 132, 131]:

$$(7.4) \quad \{q^i, q^j\}_{\tilde{\omega}_\xi^{(2)}} = 0, \quad \{\tilde{p}_j, q^i\}_{\tilde{\omega}_\xi^{(2)}} = \delta_j^i, \quad \{\tilde{p}_i, \tilde{p}_j\}_{\tilde{\omega}_\xi^{(2)}} = 0$$

for  $i, j = \overline{1, 3}$  with respect to the reference frame  $\mathcal{K}_f(t, q - q_f)$ , characterized by the phase space coordinates  $(q, \tilde{p}) \in T^*(N)$ , iff the Maxwell field equations

$$(7.5) \quad \partial F_{ij}/\partial q_k + \partial F_{jk}/\partial q_i + \partial F_{ki}/\partial q_j = 0$$

are satisfied on  $N$  for all  $i, j, k = \overline{1, 3}$  with the curvature tensor  $F_{ij}(q) := \partial A_j/\partial q^i - \partial A_i/\partial q^j$ ,  $i, j = \overline{1, 3}$ ,  $q \in N$ .

**7.2. The Lorentz type force and Maxwell electromagnetic field equations - the Lagrangian analysis.** The Poisson structure (7.4) makes it possible to describe a charged particle  $\xi \in \mathbb{R}$ , located at point  $q \in N \subset \mathbb{R}^3$ , moving with a velocity  $dq/dt := u \in T_q(N)$  with respect to the laboratory reference frame  $\mathcal{K}(t, q)$ , specified by coordinates  $(t, q) \in M^4$ , being under the electromagnetic influence of an external charged particle  $\xi_f \in \mathbb{R}$  located at point  $q_f \in N \subset \mathbb{R}^3$  and moving with respect to the same reference frame  $\mathcal{K}(t, q)$  with a velocity  $dq_f/dt := u_f \in T_{q_f}(N)$ . Really, consider a new shifted reference frame  $\mathcal{K}'_f(t', q - q_f)$  moving with respect to the reference frame  $\mathcal{K}(t, q)$  with the velocity  $u_f$ . With respect to the reference frame  $\mathcal{K}'_f(t', q - q_f)$ , specified by coordinates  $(t', q - q_f) \in M^4$ , the charged point particle  $\xi$  moves with the velocity  $u' - u'_f := dr/dt' - dr_f/dt' \in T_{q-q_f}(N)$  and, respectively, the charged particle  $\xi_f$  stays in rest. Then one can write down the standard *classical Lagrangian function* of the charged particle  $\xi$  with a mass  $m' \in \mathbb{R}_+$  subject to the reference frame  $\mathcal{K}'_f(t', q - q_f)$ :

$$(7.6) \quad \mathcal{L}_f(q, u') = \frac{m'}{2} |u' - u'_f|^2 - \xi \varphi',$$

and the suitably Lorentz transformed scalar potential  $\varphi' = \varphi/(1 + |u'_f|^2) \in C^2(N; \mathbb{R})$  is the corresponding potential energy with respect to the reference frame  $\mathcal{K}'_f(t', q - q_f)$ . On the other hand, owing to (7.6) and the Poisson brackets (7.4) the following equality for the charged particle  $\xi$  *canonical momentum* with respect to the reference frame  $\mathcal{K}'_f(t', q - q_f)$  holds:

$$(7.7) \quad \tilde{p}' := p' + \xi A'(q) = \partial \mathcal{L}_f(q, u')/\partial u',$$

or, equivalently,

$$(7.8) \quad p' + \xi A'(q) = m'(u' - u'_f),$$

expressed in the units when the light speed  $c = 1$ . Taking into account that the charged particle  $\xi$  momentum with respect to the reference frame  $\mathcal{K}(t, q)$  equals  $p' := m'u' \in T_q(N)$ , one can easily obtain from (7.8) the important relationship

$$(7.9) \quad \xi A'(q) = -m'u'_f$$

for the vector potential  $A \in C^2(N; \mathbb{E}^3)$ , which was before obtained in [127, 126, 136] and described before in Section 3.1. Taking now into account (7.6) and (7.9) one finds the following Lagrangian equation:

$$(7.10) \quad \frac{d}{dt'}[p' + \xi A'(q)] = \partial \mathcal{L}_f(q, u') / \partial q = -\xi \nabla \varphi',$$

obtained before with respect to the shifted reference frame  $\mathcal{K}'_f(t', q - q_f)$  in [127, 126] and giving rise, as the result of obvious relationships  $p' = p, A' = A$ , to the following charged point particle  $\xi$  dynamics:

$$(7.11) \quad \begin{aligned} dp/dt &= -\xi \partial A / \partial t - \xi \nabla \varphi (1 - |u_f|^2) - \xi \langle u, \nabla \rangle A = \\ &= -\xi \partial A / \partial t - \xi \nabla \varphi - \xi \langle u, \nabla \rangle A + \\ &+ \xi \nabla \langle u, A \rangle - \xi \nabla \langle u - u_f, A \rangle = \\ &= -\xi (\partial A / \partial t + \nabla \varphi) + \xi u \times (\nabla \times A) - \xi \nabla \langle u - u_f, A \rangle \end{aligned}$$

with respect to the laboratory reference frame  $\mathcal{K}(t, q)$ . Based now on (7.11) we obtain the modified Lorentz type force

$$(7.12) \quad dp/dt = \xi E + \xi u \times B - \xi \nabla \langle u - u_f, A \rangle,$$

where we put, as usually by definition,

$$(7.13) \quad E := -\partial A / \partial t - \nabla \varphi, \quad B := \nabla \times A,$$

and slightly differing from the classical [82, 39, 57, 97] Lorentz force expression

$$(7.14) \quad dp/dt = \xi E + \xi u \times B$$

by the gradient component

$$(7.15) \quad F_c := -\xi \nabla \langle u - u_f, A \rangle.$$

It is seen that the modified Lorentz type force expression (7.12) can be naturally generalized to the relativistic case if to take into account that the standard Lorenz condition

$$(7.16) \quad \partial \varphi / \partial t + \langle \nabla, A \rangle = 0$$

is imposed on the electromagnetic potential  $(\varphi, A) \in C^2(N; M^4)$ .

Really, from (7.13) one obtains that the Lorentz invariant field equation

$$(7.17) \quad \partial^2 \varphi / \partial t^2 - \Delta \varphi = \rho_f,$$

where  $\rho_f : N \rightarrow \mathcal{D}'(N)$  is a generalized density function of the external charge distribution  $\xi_f$ . Following now by the calculations from [127, 126] we can easily find from (7.17) and the charge conservation law

$$(7.18) \quad \partial \rho_f / \partial t + \langle \nabla, j_f \rangle = 0$$

the next *Lorentz invariant* equation on the vector potential  $A \in C^2(N; \mathbb{E}^3)$ :

$$(7.19) \quad \partial^2 A / \partial t^2 - \Delta A = j_f.$$

Moreover, relationships (7.13), (7.17) and (7.19) easily entail the true classical Maxwell equations

$$(7.20) \quad \begin{aligned} \nabla \times E &= -\partial B / \partial t, \quad \nabla \times B = \partial E / \partial t + j_f, \\ \langle \nabla, E \rangle &= \rho_f, \quad \langle \nabla, B \rangle = 0 \end{aligned}$$

on the electromagnetic field  $(E, B) \in C^2(N; \mathbb{E}^3 \times \mathbb{E}^3)$ .



Consider now the Lorenz condition (7.16) and observe that it is equivalent to the following local conservation law:

$$(7.21) \quad \frac{d}{dt} \int_{\Omega_t} W d^3q = 0,$$

giving rise to the important relationship for the magnetic potential  $A \in C^2(N; \mathbb{E}^3)$

$$(7.22) \quad A = u_f \varphi$$

with respect to the laboratory reference frame  $\mathcal{K}(t, q)$ , where  $\Omega_t \subset N$  is any open domain with the smooth boundary  $\partial\Omega_t$ , moving jointly with the charge distribution  $\xi_f$  in the domain  $N \subset \mathbb{R}^3$  with the corresponding velocity  $u'_f$ . Taking into account relationship (7.9) one can find the expression for our charged particle  $\xi$  “inertial” mass:

$$(7.23) \quad m = -\bar{W}, \quad \bar{W} := \xi\varphi,$$

coinciding with that obtained before in [127, 126, 136], where we denoted by  $\bar{W} \in C^2(N; \mathbb{R})$  the corresponding potential energy of the charged point particle  $\xi$ .

**7.3. The modified least action principle and the Hamiltonian analysis.** Based on the representations (7.22) and (7.23) one can rewrite the determining Lagrangian equation (7.10) with respect to the shifted reference frame  $\mathcal{K}'_f(t', q_f)$  as follows:

$$(7.24) \quad \frac{d}{dt'} [-\bar{W}'(u' - u'_f)] = -\nabla \bar{W}',$$

which is reduced to the Lorentz type force expression (7.12) calculated with respect to the reference frame  $\mathcal{K}(t, q)$ :

$$(7.25) \quad dp/dt = \xi E + \xi u \times B - \xi \nabla \langle u - u_f, A \rangle,$$

where we put, as before,

$$(7.26) \quad E := -\partial A / \partial t - \nabla \varphi, \quad B := \nabla \times A.$$

*Remark 7.1.* It is interesting to remark here that equation (7.25) does not allow the Lagrangian representation with respect to the reference frame  $\mathcal{K}(t, q)$  in contrast to that of equation (7.24) which is equivalent to (7.10).

The remark above is a challenging source of our further analysis concerning the direct relativistic generalization of the modified Lorentz type force (7.12). Namely, the following proposition holds.

**Proposition 7.2.** *The Lorentz type force (7.12) in the case when the charged point particle  $\xi$  momentum is defined, owing to (7.23), as  $p = -\bar{W}u$  is the exact relativistic expression allowing the Lagrangian representation of the charged particle  $\xi$  dynamics with respect to the rest reference frame  $\mathcal{K}_\tau(\tau, q - q_f)$ , related to the shifted reference frame  $\mathcal{K}'_f(t', q - q_f)$  by means of the classical relativistic proper time infinitesimal transformation:*

$$(7.27) \quad dt' = d\tau(1 + |u' - u'_f|^2)^{1/2},$$

where  $\tau \in \mathbb{R}$  is the proper time parameter in the rest reference frame  $\mathcal{K}_\tau(\tau, q - q_f)$ .

*Proof.* Take the following action functional with respect to the charged point particle  $\xi$  rest reference frame  $\mathcal{K}_\tau(\tau, q - q_f)$ :

$$(7.28) \quad S^{(\tau)} := - \int_{t_1(\tau_1)}^{t_2(\tau_2)} \bar{W}' dt' = \int_{\tau_1}^{\tau_2} \bar{W}' (1 + |u' - u'_f|^2)^{1/2} d\tau,$$

where the proper temporal values  $\tau_1, \tau_2 \in \mathbb{R}$  are considered, in a Feynman spirit [57], to be fixed in contrast to the temporal parameters  $t_1(\tau_1), t_2(\tau_2) \in \mathbb{R}$  depending, owing to (7.27), on the charged particle  $\xi$  trajectory in the phase space  $M^4$ . The least action condition

$$(7.29) \quad \delta S^{(\tau)} = 0, \delta q(\tau_1) = 0 = \delta q(\tau_2),$$

applied to (7.28), entails exactly the dynamical equation (7.24), being simultaneously equivalent to the relativistic Lorentz type force expression (7.12) with respect to the laboratory reference frame  $\mathcal{K}(t, q)$ . The latter proves the proposition.  $\square$

Making use of the relationships between the reference frames  $\mathcal{K}(t, q)$  and  $\mathcal{K}_\tau(\tau, q - q_f)$  in the case when the external charge particle velocity  $u_f = 0$ , we can easily derive the following corollary.

*Corollary 7.3.* Let the external charge point  $e_f$  be in rest, that is the velocity  $u_f = 0$ . Then equation (7.24) reduces to

$$(7.30) \quad \frac{d}{dt}(-\bar{W}u) = -\nabla\bar{W},$$

allowing the following conservation law:

$$(7.31) \quad H_0 = \bar{W}(1 - |u|^2)^{1/2} = -(\bar{W}^2 - |p|^2)^{1/2}.$$

Moreover, equation (7.30) is Hamiltonian with respect to the canonical Poisson structure (7.4), Hamiltonian function (7.31) and the rest reference frame  $\mathcal{K}_\tau(\tau, q)$ :

$$(7.32) \quad \left. \begin{aligned} dq/d\tau &:= \partial H_0 / \partial p = p(\bar{W}^2 - |p|^2)^{-1/2} \\ dp/d\tau &:= -\partial H_0 / \partial q = -\bar{W}(\bar{W}^2 - |p|^2)^{-1/2} \nabla\bar{W} \end{aligned} \right\} \Rightarrow \left. \begin{aligned} dq/dt &= -p\bar{W}^{-1}, \\ dp/dt &= -\nabla\bar{W} \end{aligned} \right\}.$$

In addition, if to define the rest particle mass  $m_0 := -H_0|_{u=0}$ , the "inertial" particle mass quantity  $m \in \mathbb{R}$  obtains the well known classical relativistic form

$$(7.33) \quad m = -\bar{W} = m_0(1 - |u|^2)^{-1/2},$$

depending on the particle velocity  $u \in \mathbb{R}^3$ .

Concerning the general case of equation (7.24) analogous ones to the above results hold, which were also described in part in Section 3.1. We need only to mention that the induced Hamiltonian structure of the general equation (7.24) results naturally from its least action representation (7.28) and (7.29) with respect to the rest reference frame  $\mathcal{K}_\tau(\tau, q)$ .

**7.3.1. Comments.** Within Section 7 we have demonstrated the complete legacy of the Feynman's approach to the Lorentz force based derivation of the Maxwell electromagnetic field equations. Moreover, we have succeeded in finding the exact relationship between the Feynman's approach and the vacuum field approach of Section 3.1, devised before in [127, 126]. Thus, the results obtained firmly argue for the deep physical backgrounds lying in the vacuum field theory approach, based on which one can simultaneously describe the physical phenomena both of electromagnetic and gravity origins. The latter is physically based on the particle "inertial" mass expression (7.23), naturally following both from the Feynman's approach to the Lorentz force derivation and from the vacuum field approach.

## 8. THE GENERALIZED FOCK SPACES, QUANTUM CURRENTS ALGEBRA REPRESENTATIONS AND ELECTRODYNAMICS

**8.1. Preliminaries: Fock space and its realizations.** Let  $\Phi$  be a separable Hilbert space,  $F$  be a topological real linear space and  $\mathcal{A} := \{A(f) : f \in F\}$  a family of commuting self-adjoint operators in  $\Phi$  (i.e. these operators commute in the sense of their resolutions of the identity). Consider the Gelfand rigging [12] of the Hilbert space  $\Phi$ , i.e., a chain

$$(8.1) \quad \mathcal{D} \subset \Phi_+ \subset \Phi \subset \Phi_- \subset \mathcal{D}'$$

in which  $\Phi_+$  and  $\Phi_-$  are further Hilbert spaces, and the inclusions are dense and continuous, i.e.  $\Phi_+$  is topologically (densely and continuously) and quasi-nucleus (the inclusion operator  $i : \Phi_+ \rightarrow \Phi$  is of the Hilbert - Schmidt type) embedded into  $\Phi$ , the space  $\Phi_-$  is the dual of  $\Phi_+$  with respect to the scalar product  $\langle \cdot, \cdot \rangle_\Phi$  in  $\Phi$ , and  $\mathcal{D}$  is a separable projective limit of Hilbert spaces, topologically embedded into  $\Phi_+$ . Then, the following structural theorem [12, 14] holds:

**Theorem 8.1.** Assume that the family of operators  $\mathcal{A}$  satisfies the following conditions:

- a)  $\mathcal{D} \subset \text{Dom}A(f)$ ,  $f \in F$ , and the closure of the operator  $A(f) \upharpoonright \mathcal{D}$  coincides with  $A(f)$  for any  $f \in F$ , that is  $A(f) \upharpoonright \mathcal{D} = A(f)$  in  $\Phi$ ;
- b) the Range  $A(f) \upharpoonright \mathcal{D} \subset \Phi_+$  for any  $f \in F$ ;
- c) for every  $\psi \in \mathcal{D}$  the mapping  $F \ni f \rightarrow A(f)\psi \in \Phi_+$  is linear and continuous;
- d) there exists a strong cyclic (vacuum) vector  $|\Omega\rangle \in \bigcap_{f \in F} \text{Dom}A(f)$ , such that the set of all vectors  $|\Omega\rangle, \prod_{j=1}^n A(f_j)|\Omega\rangle, n \in \mathbb{Z}_+,$  is total in  $\Phi_+$  (i.e. their linear hull is dense in  $\Phi_+$ ).

Then there exists a probability measure  $\mu$  on  $(F', C_\sigma(F'))$ , where  $F'$  is the dual of  $F$  and  $C_\sigma(F')$  is the  $\sigma$ -algebra generated by cylinder sets in  $F'$  such that, for  $\mu$ -almost every  $\eta \in F'$  there is

a generalized joint eigenvector  $\omega(\eta) \in \Phi_-$  of the family  $\mathcal{A}$ , corresponding to the joint eigenvalue  $\eta \in F'$ , that is

$$(8.2) \quad \langle \omega(\eta), A(f)\psi \rangle_{\Phi} = \eta(f) \langle \omega(\eta), \psi \rangle_{\Phi}$$

with  $\eta(f) \in \mathbb{R}$  denoting the pairing between  $F$  and  $F'$ .

The mapping

$$(8.3) \quad \Phi_+ \ni \psi \longrightarrow \langle \omega(\eta), \psi \rangle_{\Phi} := \psi(\eta) \in \mathbb{C}$$

for any  $\eta \in F'$  can be continuously extended to a unitary surjective operator  $\mathcal{F} : \Phi \longrightarrow L_2^{(\mu)}(F'; \mathbb{C})$ , where

$$(8.4) \quad \mathcal{F} \psi(\eta) := \psi(\eta)$$

for any  $\eta \in F'$  is a generalized Fourier transform, corresponding to the family  $\mathcal{A}$ . Moreover, the image of the operator  $A(f)$ ,  $f \in F'$ , under the  $\mathcal{F}$ -mapping is the operator of multiplication by the function  $F' \ni \eta \rightarrow \eta(f) \in \mathbb{C}$ .

We assume additionally that the main Hilbert space  $\Phi$  possesses the standard Fock space (bose)-structure [21, 12, 129], that is

$$(8.5) \quad \Phi = \oplus_{n \in \mathbb{Z}_+} \Phi_{(s)}^{\otimes n},$$

where subspaces  $\Phi_{(s)}^{\otimes n}$ ,  $n \in \mathbb{Z}_+$ , are the symmetrized tensor products of a Hilbert space  $\mathcal{H} := L_2(\mathbb{R}^m; \mathbb{C})$ . If a vector  $g := (g_0, g_1, \dots, g_n, \dots) \in \Phi$ , its norm

$$(8.6) \quad \|g\|_{\Phi} := \left( \sum_{n \in \mathbb{Z}_+} \|g_n\|_n^2 \right)^{1/2},$$

where  $g_n \in \Phi_{(s)}^{\otimes n} \simeq L_{2,(s)}((\mathbb{R}^m)^{\otimes n}; \mathbb{C})$  and  $\| \dots \|_n$  is the corresponding norm in  $\Phi_{(s)}^{\otimes n}$  for all  $n \in \mathbb{Z}_+$ . Note here that concerning the rigging structure (8.1), there holds the corresponding rigging for the Hilbert spaces  $\Phi_{(s)}^{\otimes n}$ ,  $n \in \mathbb{Z}_+$ , that is

$$(8.7) \quad \mathcal{D}_{(s)}^n \subset \Phi_{(s),+}^{\otimes n} \subset \Phi_{(s)}^{\otimes n} \subset \Phi_{(s),-}^{\otimes n}$$

with some suitably chosen dense and separable topological spaces of symmetric functions  $\mathcal{D}_{(s)}^n$ ,  $n \in \mathbb{Z}_+$ . Concerning expansion (8.1) we obtain by means of projective and inductive limits [15, 17, 12, 14] the quasi-nucleus rigging of the Fock space  $\Phi$  in the form (8.1):

$$\mathcal{D} \subset \Phi_+ \subset \Phi \subset \Phi_- \subset \mathcal{D}'.$$

Consider now any vector  $|(\alpha)_n\rangle \in \Phi_{(s)}^{\otimes n}$ ,  $n \in \mathbb{Z}_+$ , which can be written [15, 21, 93] in the following canonical Dirac ket-form:

$$(8.8) \quad |(\alpha)_n\rangle := |\alpha_1, \alpha_2, \dots, \alpha_n\rangle,$$

where, by definition,

$$(8.9) \quad |\alpha_1, \alpha_2, \dots, \alpha_n\rangle := \frac{1}{\sqrt{n!}} \sum_{\sigma \in S_n} |\alpha_{\sigma(1)}\rangle \otimes |\alpha_{\sigma(2)}\rangle \dots |\alpha_{\sigma(n)}\rangle$$

and  $|\alpha_j\rangle \in \Phi_{(s)}^{\otimes 1}(\mathbb{R}^m; \mathbb{C}) := \mathcal{H}$  for any fixed  $j \in \mathbb{Z}_+$ . The corresponding scalar product of base vectors as (8.9) is given as follows:

$$(8.10) \quad \begin{aligned} \langle (\beta)_n | (\alpha)_n \rangle &:= \langle \beta_n, \beta_{n-1}, \dots, \beta_2, \beta_1 | \alpha_1, \alpha_2, \dots, \alpha_{n-1}, \alpha_n \rangle \\ &= \sum_{\sigma \in S_n} \langle \beta_1 | \alpha_{\sigma(1)} \rangle \dots \langle \beta_n | \alpha_{\sigma(n)} \rangle := \text{per} \{ \langle \beta_i | \alpha_j \rangle : i, j = \overline{1, n} \}, \end{aligned}$$

where “per” denotes the permanent of matrix and  $\langle \cdot | \cdot \rangle$  is the corresponding product in the Hilbert space  $\mathcal{H}$ . Based now on representation (8.8) one can define an operator  $a^+(\alpha) : \Phi_{(s)}^{\otimes n} \longrightarrow \Phi_{(s)}^{\otimes (n+1)}$  for any  $|\alpha\rangle \in \mathcal{H}$  as follows:

$$(8.11) \quad a^+(\alpha) |\alpha_1, \alpha_2, \dots, \alpha_n\rangle := |\alpha, \alpha_1, \alpha_2, \dots, \alpha_n\rangle,$$

which is called the "creation" operator in the Fock space  $\Phi$ . The adjoint operator  $a(\beta) := (a^+(\beta))^* : \Phi_{(s)}^{\otimes(n+1)} \longrightarrow \Phi_{(s)}^{\otimes n}$  with respect to the Fock space  $\Phi$  (8.1) for any  $|\beta\rangle \in \mathcal{H}$ , called the "annihilation" operator, acts as follows:

$$(8.12) \quad a(\beta)|\alpha_1, \alpha_2, \dots, \alpha_{n+1}\rangle := \sum_{\sigma \in S_n} \langle \beta, \alpha_j \rangle |\alpha_1, \alpha_2, \dots, \alpha_{j-1}, \hat{\alpha}_j, \alpha_{j+1}, \dots, \alpha_{n+1}\rangle,$$

where the "hat" over a vector denotes that it should be omitted from the sequence.

It is easy to check that the commutator relationship

$$(8.13) \quad [a(\alpha), a^+(\beta)] = \langle \alpha, \beta \rangle$$

holds for any vectors  $|\alpha\rangle \in \mathcal{H}$  and  $|\beta\rangle \in \mathcal{H}$ . Expression (8.13), owing to the rigged structure (8.1), can be naturally extended to the general case, when vectors  $|\alpha\rangle$  and  $|\beta\rangle \in \mathcal{H}_-$ , conserving its form. In particular, taking  $|\alpha\rangle := |\alpha(x)\rangle = \frac{1}{\sqrt{2\pi}} e^{i\langle \lambda, x \rangle} \in \mathcal{H}_- := L_{2,-}(\mathbb{R}^m; \mathbb{C})$  for any  $x \in \mathbb{R}^m$ , one easily gets from (8.13) that

$$(8.14) \quad [a(x), a^+(y)] = \delta(x - y),$$

where we put, by definition,  $a^+(x) := a^+(\alpha(x))$  and  $a(y) := a(\alpha(y))$  for all  $x, y \in \mathbb{R}^m$  and denoted by  $\delta(\cdot)$  the classical Dirac delta-function.

The construction above makes it possible to observe easily that there exists a unique vacuum vector  $|\Omega\rangle \in \mathcal{H}_+$ , such that for any  $x \in \mathbb{R}^m$

$$(8.15) \quad a(x)|\Omega\rangle = 0,$$

and the set of vectors

$$(8.16) \quad \left( \prod_{j=1}^n a^+(x_j) \right) |\Omega\rangle \in \Phi_{(s)}^{\otimes n}$$

is total in  $\Phi_{(s)}^{\otimes n}$ , that is their linear integral hull over the dual functional spaces  $\hat{\Phi}_{(s)}^{\otimes n}$  is dense in the Hilbert space  $\Phi_{(s)}^{\otimes n}$  for every  $n \in \mathbb{Z}_+$ . This means that for any vector  $g \in \Phi$  the following representation

$$(8.17) \quad g = \oplus_{n \in \mathbb{Z}_+} \int_{(\mathbb{R}^m)^n} \hat{g}_n(x_1, \dots, x_n) a^+(x_1) a^+(x_2) \dots a^+(x_n) |\Omega\rangle$$

holds with the Fourier type coefficients  $\hat{g}_n \in \hat{\Phi}_{(s)}^{\otimes n}$  for all  $n \in \mathbb{Z}_+$ , with  $\hat{\Phi}_{(s)}^{\otimes 1} := \mathcal{H} \simeq L_2(\mathbb{R}^m; \mathbb{C})$ . The latter is naturally endowed with the Gelfand type quasi-nucleus rigging dual to

$$(8.18) \quad \mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_-,$$

making it possible to construct a quasi-nucleolus rigging of the dual Fock space  $\hat{\Phi} := \oplus_{n \in \mathbb{Z}_+} \hat{\Phi}_{(s)}^{\otimes n}$ . Thereby, chain (8.18) generates the dual Fock space quasi-nucleolus rigging

$$(8.19) \quad \hat{\mathcal{D}} \subset \hat{\Phi}_+ \subset \hat{\Phi} \subset \hat{\Phi}_- \subset \hat{\mathcal{D}}'$$

with respect to the central Fock type Hilbert space  $\hat{\Phi}$ , where  $\hat{\mathcal{D}} \simeq \mathcal{D}$ , easily following from (8.1) and (8.18).

Construct now the following self-adjoint operator  $\rho(x) : \Phi \rightarrow \Phi$  as

$$(8.20) \quad \rho(x) := a^+(x)a(x),$$

called the density operator at a point  $x \in \mathbb{R}^m$ , satisfying the commutation properties:

$$(8.21) \quad \begin{aligned} [\rho(x), \rho(y)] &= 0, \\ [\rho(x), a(y)] &= -a(y)\delta(x - y), \\ [\rho(x), a^+(y)] &= a^+(y)\delta(x - y) \end{aligned}$$

for all  $y \in \mathbb{R}^m$ .

Now, if to construct the following self-adjoint family  $\mathcal{A} := \left\{ \int_{\mathbb{R}^m} \rho(x) f(x) dx : f \in F \right\}$  of linear operators in the Fock space  $\Phi$ , where  $F := \mathcal{S}(\mathbb{R}^m; \mathbb{R})$  is the Schwartz functional space, one can

derive, making use of Theorem 8.1, that there exists the generalized Fourier transform (8.4), such that

$$(8.22) \quad \Phi(\mathcal{H}) = L_2^{(\mu)}(\mathcal{S}'; \mathbb{C}) \simeq \int_{\mathcal{S}'}^{\oplus} \Phi_{\eta} d\mu(\eta)$$

for some Hilbert space sets  $\Phi_{\eta}$ ,  $\eta \in F'$ , and a suitable measure  $\mu$  on  $\mathcal{S}'$ , with respect to which the corresponding joint eigenvector  $\omega(\eta) \in \Phi_+$  for any  $\eta \in F'$  generates the Fourier transformed family  $\hat{u} = \{\eta(f) \in \mathbb{R} : f \in F\}$ . Moreover, if  $\dim \Phi_{\eta} = 1$  for all  $\eta \in F$ , the Fourier transformed eigenvector  $\hat{\omega}(\eta) := \Omega(\eta) = 1$  for all  $\eta \in F'$ .

Now we will consider the family of self-adjoint operators  $\hat{u}$  as generating a unitary family  $\mathcal{U} := \{U(f) : f \in F\} = \exp(i\hat{u})$ , where for any  $\rho(f) \in \hat{u}$ ,  $f \in F$ , the operator

$$(8.23) \quad U(f) := \exp[i\rho(f)]$$

is unitary, satisfying the abelian commutation condition

$$(8.24) \quad U(f_1)U(f_2) = U(f_1 + f_2)$$

for any  $f_1, f_2 \in F$ .

Since, in general, the unitary family  $\mathcal{U} = \exp(i\hat{u})$  is defined in some Hilbert space  $\Phi$ , not necessarily being of Fock type, the important problem of describing its Hilbertian cyclic representation spaces arises, within which the factorization

$$(8.25) \quad \rho(f) = \int_{\mathbb{R}^m} a^+(x)a(x)f(x)dx$$

jointly with relationships (8.21) hold for any  $f \in F$ . This problem can be treated using mathematical tools devised both within the representation theory of  $C^*$ -algebras [40, 42] and the Gelfand–Vilenkin [61] approach. Below we will describe the main features of the Gelfand–Vilenkin formalism, being much more suitable for the task, providing a reasonably unified framework of constructing the corresponding representations.

**Definition 8.2.** Let  $F$  be a locally convex topological vector space,  $F_0 \subset F$  be a finite dimensional subspace of  $F$ . Let  $F^0 \subseteq F'$  be defined by

$$(8.26) \quad F^0 := \{\xi \in F' : \xi|_{F_0} = 0\},$$

and called the annihilator of  $F_0$ .

The quotient space  $F'^0 := F'/F^0$  may be identified with  $F'_0 \subset F'$ , the adjoint space of  $F_0$ .

**Definition 8.3.** Let  $A \subseteq F'$ ; then the subset

$$(8.27) \quad X_{F^0}^{(A)} := \{\xi \in F' : \xi + F^0 \subset A\}$$

is called the cylinder set with base  $A$  and generating subspace  $F^0$ .

**Definition 8.4.** Let  $n = \dim F_0 = \dim F'_0 = \dim F'^0$ . One says that a cylinder set  $X^{(A)}$  has Borel base, if  $A$  is Borel, when regarded as a subset of  $\mathbb{R}^m$ .

The family of cylinder sets with Borel base forms an algebra of sets.

**Definition 8.5.** The measurable sets in  $F'$  are the elements of the  $\sigma$ -algebra generated by the cylinder sets with Borel base.

**Definition 8.6.** A cylindrical measure in  $F'$  is a real-valued  $\sigma$ -pre-additive function  $\mu$  defined on the algebra of cylinder sets with Borel base and satisfying the conditions  $0 \leq \mu(X) \leq 1$  for any  $X$ ,  $\mu(F') = 1$  and  $\mu\left(\coprod_{j \in \mathbb{Z}_+} X_j\right) = \sum_{j \in \mathbb{Z}_+} \mu(X_j)$ , if all sets  $X_j \subset F'$ ,  $j \in \mathbb{Z}_+$ , have a common generating subspace  $F_0 \subset F$ .

**Definition 8.7.** A cylindrical measure  $\mu$  satisfies the commutativity condition if and only if for any bounded continuous function  $\alpha : F^n \rightarrow \mathbb{R}$  of  $n \in \mathbb{Z}_+$  real variables the function

$$(8.28) \quad \alpha[f_1, f_2, \dots, f_n] := \int_{F'} \alpha(\eta(f_1), \eta(f_2), \dots, \eta(f_n)) d\mu(\eta)$$

is sequentially continuous in  $f_j \in F$ ,  $j = \overline{1, m}$ . (It is well known [61, 65] that in countably normalized spaces the properties of sequential and ordinary continuity are equivalent).

**Definition 8.8.** A cylindrical measure  $\mu$  is countably additive if and only if for any cylinder set  $X = \coprod_{j \in \mathbb{Z}_+} X_j$ , which is the union of countably many mutually disjoint cylinder sets  $X_j \subset F', j \in \mathbb{Z}_+$ ,  $\mu(X) = \sum_{j \in \mathbb{Z}_+} \mu(X_j)$ .

The following propositions hold.

**Proposition 8.9.** A countably additive cylindrical measure  $\mu$  can be extended to a countably additive measure on the  $\sigma$ -algebra, generated by the cylinder sets with Borel base. Such a measure will also be called a cylindrical measure.

**Proposition 8.10.** Let  $F$  be a nuclear space. Then any cylindrical measure  $\mu$  on  $F'$ , satisfying the continuity condition, is countably additive.

**Definition 8.11.** Let  $\mu$  be a cylindrical measure in  $F'$ . The Fourier transform of  $\mu$  is the nonlinear functional

$$(8.29) \quad \mathcal{L}(f) := \int_{F'} \exp[i\eta(f)] d\mu(\eta).$$

**Definition 8.12.** The nonlinear functional  $\mathcal{L} : F \rightarrow \mathbb{C}$  on  $F$ , defined by (8.29), is called positive definite, if and only if for all  $f_j \in F$  and  $\lambda_j \in \mathbb{C}$ ,  $j = \overline{1, n}$ , the condition

$$(8.30) \quad \sum_{j,k=1}^n \bar{\lambda}_j \mathcal{L}(f_k - f_j) \lambda_k \geq 0$$

holds for any  $n \in \mathbb{Z}_+$ .

**Proposition 8.13.** The functional  $\mathcal{L} : F \rightarrow \mathbb{C}$  on  $F$ , defined by (8.29), is the Fourier transform of a cylindrical measure on  $F'$ , if and only if it is positive definite, sequentially continuous and satisfying the condition  $\mathcal{L}(0) = 1$ .

Suppose now that we have a continuous unitary representation of the unitary family  $\mathcal{U}$  in a Hilbert space  $\Phi$  with a cyclic vector  $|\Omega\rangle \in \Phi$ . Then we can put

$$(8.31) \quad \mathcal{L}(f) := \langle \Omega | U(f) | \Omega \rangle$$

for any  $f \in F := \mathcal{S}$ , being the Schwartz space on  $\mathbb{R}^m$ , and observe that functional (8.31) is continuous on  $F$  owing to the continuity of the representation. Therefore, this functional is the generalized Fourier transform of a cylindrical measure  $\mu$  on  $\mathcal{S}'$ :

$$(8.32) \quad \langle \Omega | U(f) | \Omega \rangle = \int_{\mathcal{S}'} \exp[i\eta(f)] d\mu(\eta).$$

From the spectral point of view, based on Theorem 8.1, there is an isomorphism between the Hilbert spaces  $\Phi$  and  $L_2^{(\mu)}(\mathcal{S}'; \mathbb{C})$ , defined by  $|\Omega\rangle \rightarrow \Omega(\eta) = 1$  and  $U(f)|\Omega\rangle \rightarrow \exp[i\eta(f)]$  and next extended by linearity upon the whole Hilbert space  $\Phi$ .

In the case of the non-cyclic case there exists a finite or countably infinite family of measures  $\{\mu_k : k \in \mathbb{Z}_+\}$  on  $\mathcal{S}'$ , with  $\Phi \simeq \oplus_{k \in \mathbb{Z}_+} L_2^{(\mu_k)}(\mathcal{S}'; \mathbb{C})$  and the unitary operator  $U(f) : \Phi \rightarrow \Phi$  for any  $f \in \mathcal{S}'$  corresponds in all  $L_2^{(\mu_k)}(\mathcal{S}'; \mathbb{C})$ ,  $k \in \mathbb{Z}_+$ , to  $\exp[i\eta(f)]$ . This means that there exists a single cylindrical measure  $\mu$  on  $\mathcal{S}'$  and a  $\mu$ -measurable field of Hilbert spaces  $\Phi_\eta$  on  $\mathcal{S}'$ , such that

$$(8.33) \quad \Phi \simeq \int_{\mathcal{S}'}^\oplus \Phi_\eta d\mu(\eta),$$

with  $U(f) : \Phi \rightarrow \Phi$ , corresponding [61] to the operator of multiplication by  $\exp[i\eta(f)]$  for any  $f \in \mathcal{S}$  and  $\eta \in \mathcal{S}'$ . Thereby, having constructed the nonlinear functional (8.29) in an exact analytical form, one can retrieve the representation of the unitary family  $\mathcal{U}$  in the corresponding Hilbert space  $\Phi$  of the Fock type, making use of the suitable factorization (8.25) as follows:  $\Phi = \oplus_{n \in \mathbb{Z}_+} \Phi_n$ , where

$$(8.34) \quad \Phi_n = \text{span}_{f_n \in L_{2,s}(\mathbb{R}^m \times n; \mathbb{C})} \left\{ \prod_{j=\overline{1,n}} a^+(x_j) |\Omega\rangle \right\},$$

for all  $n \in \mathbb{Z}_+$ . The cyclic vector  $|\Omega\rangle \in \Phi$  can be, in particular, obtained as the ground state vector of some unbounded self-adjoint positive definite Hamilton operator  $\mathbb{H} : \Phi \longrightarrow \Phi$ , commuting with the self-adjoint particles number operator

$$(8.35) \quad \mathbb{N} := \int_{\mathbb{R}^m} \rho(x) dx,$$

that is  $[\mathbb{H}, \mathbb{N}] = 0$ . Moreover, the conditions

$$(8.36) \quad \mathbb{H}|\Omega\rangle = 0$$

and

$$(8.37) \quad \inf_{g \in \text{dom } \mathbb{H}} \langle g, \mathbb{H}g \rangle = \langle \Omega | \mathbb{H} | \Omega \rangle = 0$$

hold for the operator  $\mathbb{H} : \Phi \longrightarrow \Phi$ , where  $\text{dom } \mathbb{H}$  denotes its domain of definition.

To find the functional (8.31), which is called the generating Bogolubov type functional for moment distribution functions

$$(8.38) \quad F_n(x_1, x_2, \dots, x_n) := \langle \Omega | : \rho(x_1) \rho(x_2) \dots \rho(x_n) : | \Omega \rangle,$$

where  $x_j \in \mathbb{R}^m$ ,  $j = \overline{1, n}$ , and the normal ordering operation  $: \cdot :$  is defined as

$$(8.39) \quad : \rho(x_1) \rho(x_2) \dots \rho(x_n) : = \prod_{j=1}^n \left( \rho(x_j) - \sum_{k=1}^j \delta(x_j - x_k) \right),$$

it is convenient to choose the Hamilton operator  $\mathbb{H} : \Phi \longrightarrow \Phi$  in the following [66, 65, 25] algebraic form:

$$(8.40) \quad \mathbb{H} := \frac{1}{2} \int_{\mathbb{R}^m} K^+(x) \rho^{-1}(x) K(x) dx + V(\rho),$$

being equivalent in the Hilbert space  $\Phi$  to the positive definite operator expression

$$(8.41) \quad \mathbb{H} := \frac{1}{2} \int_{\mathbb{R}^m} (K^+(x) - A(x; \rho)) \rho^{-1}(x) (K(x) - A(x; \rho)) dx,$$

satisfying conditions (8.36) and (8.37), where  $A(x; \rho) : \Phi \rightarrow \Phi$ ,  $x \in \mathbb{R}^m$ , is some specially chosen linear self-adjoint operator. The “potential” operator  $V(\rho) : \Phi \longrightarrow \Phi$  is, in general, a polynomial (or analytical) functional of the density operator  $\rho(x) : \Phi \longrightarrow \Phi$  and the operator is given as

$$(8.42) \quad K(x) := \nabla_x \rho(x) / 2 + iJ(x),$$

where the self-adjoint “current” operator  $J(x) : \Phi \longrightarrow \Phi$  can be defined (but non-uniquely) from the equality

$$(8.43) \quad \partial \rho / \partial t = \frac{1}{i} [\mathbb{H}, \rho(x)] = - \langle \nabla_x, J(x) \rangle,$$

holding for all  $x \in \mathbb{R}^m$ . Such an operator  $J(x) : \Phi \longrightarrow \Phi$ ,  $x \in \mathbb{R}^m$ , can exist owing to the commutation condition  $[\mathbb{H}, \mathbb{N}] = 0$ , giving rise to the continuity relationship (8.43), if taking into account that supports  $\text{supp } \rho$  of the density operator  $\rho(x) : \Phi \longrightarrow \Phi$ ,  $x \in \mathbb{R}^m$ , can be chosen arbitrarily owing to the independence of (8.43) on the potential operator  $V(\rho) : \Phi \longrightarrow \Phi$ , but its strict dependence on the corresponding representation (8.33). In particular, based on the Fock space  $\Phi$ , defined by (8.1) and generated by the creation-annihilation operators (8.11) and (8.12), the current operator  $J(x) : \Phi \longrightarrow \Phi$ ,  $x \in \mathbb{R}^m$ , can be constructed as follows:

$$(8.44) \quad J(x) = \frac{1}{2i} [a^+(x) \nabla a(x) - \nabla a^+(x) a(x)],$$

satisfying jointly with the density operator  $\rho(x) : \Phi \longrightarrow \Phi$ ,  $x \in \mathbb{R}^m$ , defined by (8.20), the following quantum current Lie algebra [67, 25, 119] relationships:

$$(8.45) \quad \begin{aligned} [J(g_1), J(g_2)] &= iJ([g_1, g_2]), \\ [J(g_1), \rho(f_1)] &= i\rho(\langle g_1, \nabla f_1 \rangle), \\ [\rho(f_1), \rho(f_2)] &= 0, \end{aligned}$$

holding for all  $f_1, f_2 \in F$  and  $g_1, g_2 \in F^3$ , where we put, by definition,

$$(8.46) \quad [g_1, g_2] := \langle g_1, \nabla \rangle g_2 - \langle g_2, \nabla \rangle g_1,$$

being the usual commutator of vector fields in the Euclidean space  $\mathbb{E}^m$ . It is easy to observe that the current algebra (8.45) is the Lie algebra  $\mathcal{G}$ , corresponding to the Banach Lie group  $G = \text{Diff } \mathbb{E}^3 \ltimes F$ , the semidirect product of the Banach Lie group of diffeomorphisms  $\text{Diff } \mathbb{E}^3$  of the three-dimensional Euclidean space  $\mathbb{E}^3$  and the abelian subject to the multiplicative operation Banach group of smooth functions  $F$ . We note also that representation (8.41) holds only under the condition that there exists such a self-adjoint operator  $\mathcal{A}(x; \rho) : \Phi \longrightarrow \Phi$ ,  $x \in \mathbb{R}^m$ , that

$$(8.47) \quad K(x)|\Omega\rangle = \mathcal{A}(x; \rho)|\Omega\rangle$$

for all ground states  $|\Omega\rangle \in \Phi$ , correspond to suitably chosen potential operators  $V(\rho) : \Phi \longrightarrow \Phi$ .

The self-adjointness of the operator  $\mathcal{A}(x; \rho) : \Phi \longrightarrow \Phi$ ,  $x \in \mathbb{R}^m$ , can be stated following schemes from works [66, 25], under the additional condition of the existence of such a linear anti-unitary mapping  $T : \Phi \longrightarrow \Phi$  that the following invariance conditions hold:

$$(8.48) \quad T\rho(x)T^{-1} = \rho(x), \quad T J(x) T^{-1} = -J(x), \quad T|\Omega\rangle = |\Omega\rangle$$

for any  $x \in \mathbb{R}^m$ . Thereby, owing to conditions (8.48), the following expressions

$$(8.49) \quad K^*(x)|\Omega\rangle = \mathcal{A}(x; \rho)|\Omega\rangle = K(x)|\Omega\rangle$$

hold for any  $x \in \mathbb{R}^m$ , giving rise to the self-adjointness of the operator  $\mathcal{A}(x; \rho) : \Phi \longrightarrow \Phi$ ,  $x \in \mathbb{R}^m$ .

Based now on the construction above one easily deduces from expression (8.43) that the generating Bogolubov type functional (8.31) obeys for all  $x \in \mathbb{R}^m$  the following functional-differential equation:

$$(8.50) \quad [\nabla_x - i\nabla_x f] \frac{1}{2i} \frac{\delta \mathcal{L}(f)}{\delta f(x)} = \mathcal{A} \left( x; \frac{1}{i} \frac{\delta}{\delta f} \right) \mathcal{L}(f),$$

whose solutions should satisfy the Fourier transform representation (8.32). In particular, a wide class of special so-called Poissonian white noise type solutions to the functional-differential equation (8.50) was obtained in [66, 25] by means of functional-operator methods in the following generalized form:

$$(8.51) \quad \mathcal{L}(f) = \exp \left\{ 2\mathcal{A} \left( \frac{1}{i} \frac{\delta}{\delta f} \right) \right\} \exp \left( \bar{\rho} \int_{\mathbb{R}^m} \{ \exp[if(x)] - 1 \} dx \right),$$

where  $\bar{\rho} := \langle \Omega | \rho | \Omega \rangle \in \mathbb{R}_+$  is a Poisson distribution density parameter.

It is worth to remark here that solutions to equation (8.50) realize the suitable physically motivated representations of the abelian Banach subgroup  $F$  of the Banach group  $G = \text{Diff } \mathbb{E}^3 \ltimes F$ , mentioned above. In the general case of the Banach group  $G = \text{Diff } \mathbb{E}^3 \ltimes F$  one can also construct [25, 128] a generalized Bogolubov type functional equation, whose solutions give rise to suitable physically motivated representations of the corresponding current Lie algebra  $\mathcal{G}$ .

**8.2. The quantum current Lie algebra and the magnetic Aharonow-Bohm effect.** In the Section above we could get convinced that different representations of the equal-time current algebra (8.45)

$$(8.52) \quad \begin{aligned} [\rho(f_1), \rho(f_2)] &= 0, \\ [J(g_1), \rho(f_1)] &= i\rho(\langle g_1, \nabla f_1 \rangle), \\ [J(g_1), J(g_2)] &= iJ([g_2, g_1]) + i\rho(\langle B, g_1 \times g_2 \rangle), \end{aligned}$$

where  $f_1, f_2 \in F$  and  $g_2, g_1 \in F^3$ , acting in the Fock space  $\Phi$  and describing a non-relativistic quasi-stationary system consisting of a test charged particle  $q$ , imbedded into a cylindrical region  $\Gamma \subset \mathbb{E}^3$ , being under influence of the magnetic field  $B = \nabla \times A$ . Here  $A \in \mathbb{E}^3$  is a magnetic vector potential, the sign “ $\times$ ” means the vector product in  $\mathbb{E}^3$  and the current  $J(x) : \Phi \longrightarrow \Phi$ ,  $x \in \mathbb{R}^m$ , is defined, owing to the *minimal interaction* principle (8.17), as

$$(8.53) \quad J(x) := \frac{1}{2} a^+(x) \left( \frac{1}{i} \nabla - A \right) a(x) - \left[ \left( \frac{1}{i} \nabla + A \right) a^+(x) \right] a(x).$$

In particular, it is assumed that  $\text{supp } B \subset \Gamma$  that gives rise to the equality  $\rho(\langle B, g_1 \times g_2 \rangle) = 0$  for all points  $x \in \mathbb{E}^3 \setminus \Gamma$ .

As the suitable representations of the current algebra  $\mathcal{G}$ , defined by (8.53), describe the physical quantum states of the system  $\Gamma$  under regards, we consider them following [67, 68] as those realized



in the Hilbert space  $L_2(\mathbb{E}^3; \mathbb{C})$  under the condition that the charged test particle  $q$  can penetrate the boundary  $\partial\Gamma$  of the region  $\Gamma$ . Namely, let for  $\psi \in L_2(\mathbb{E}^3; \mathbb{C})$

$$(8.54) \quad \begin{aligned} \rho(f)\psi(x) &: = f(x)\psi(x), \\ J(g)\psi(x) &= \frac{1}{2i} \left\{ [ \langle g(x), \cdot \nabla \rangle + \langle \nabla, \cdot g(x) \rangle ] - \langle g(x), \int_{\Gamma} d^3y \frac{\nabla \times B(y)}{4\pi|x-y|} \rangle \right\} \psi(x), \end{aligned}$$

for all  $x \in \mathbb{E}^3$ , where the sign “ $\cdot$ ” means that the natural operator composition. When deriving (8.54) there was imposed the invariant Coulomb gauge constraint  $\langle \nabla, A \rangle = 0$  allowing to determine the vector potential  $A \in \mathbb{E}^3$  as

$$(8.55) \quad A = \int_{\Gamma} d^3y \frac{\nabla \times B(y)}{4\pi|x-y|}$$

using the classical Maxwell equations (1.6), since the electric displacement current component  $\partial E / \partial t = 0$ . The wave function  $\psi \in L_2(\mathbb{E}^3; \mathbb{C})$  satisfies in the cylindrical coordinates  $x(r, \theta, z) \in \Gamma$  the natural quasi-periodical condition

$$(8.56) \quad \psi(r, \theta + 2\pi n, z) = \exp(i\lambda n) \psi(r, \theta, z)$$

for some  $\lambda \in \mathbb{R}$  and any  $n \in \mathbb{Z}$ , which should be determined from the physically realizable representation (8.54). To do this, we need preliminarily to define the following [68] unitary operator in the Hilbert space  $L_2(\mathbb{E}^3; \mathbb{C})$ :

$$(8.57) \quad Q\psi(x) := \begin{cases} \psi(x), & x \in \Gamma; \\ \exp \left\{ -i \int_{l_{\infty}}^x dl(y) \int_{\mathbb{E}^3 \setminus \Gamma} dy' \frac{\nabla \times B(y')}{|y-y'|} \right\} \psi(x), & x \in \mathbb{E}^3 \setminus \Gamma; \end{cases}$$

where the path  $l_{\infty} \subset \mathbb{E}^3 \setminus \Gamma$  connects an infinite point  $\infty \in \mathbb{E}^3$  with the chosen point  $x \in \mathbb{E}^3 \setminus \Gamma$ . Making use now of the unitary transformation  $\tilde{J}(g) := QJ(g)Q^{-1}$  and the fact that the magnetic field

$$(8.58) \quad B(x) = \nabla \times \int_{\Gamma} \frac{\nabla \times B(y) d^3y}{4\pi|x-y|}$$

for any  $x \in \mathbb{E}^3$ , one obtains easily that the current operator  $\tilde{J}(g) : L_2(\mathbb{E}^3; \mathbb{C}) \rightarrow L_2(\mathbb{E}^3; \mathbb{C})$  is self-adjoint for any  $g \in F^3$  and

$$(8.59) \quad \tilde{J}(g)\psi(x) = \frac{1}{2i} [ \langle g(x), \cdot \nabla \rangle + \langle \nabla, \cdot g(x) \rangle ] \psi(x),$$

and whose domain of definition  $\text{dom } \tilde{J}(g) \subset L_2(\mathbb{E}^3; \mathbb{C})$  is constrained by the functions  $\psi \in L_2(\mathbb{E}^3; \mathbb{C})$ , satisfying the condition

$$(8.60) \quad \psi(r, \theta + 2\pi, z) = \exp[i\lambda(B)] \psi(r, \theta, z),$$

where, owing to (8.57),

$$(8.61) \quad \lambda(B) = - \int_{\partial\Gamma} \langle B, dS \rangle.$$

Thus, the found above representation (8.59) of the current Lie algebra (8.52) in the Hilbert space  $L_2(\mathbb{E}^3; \mathbb{C})$ , in the case when  $\text{supp } B \subset \Gamma$ , describes the complete set of observables if the charged particle  $q$  is not excluded from the region  $\Gamma$ . In contrast, if the region  $\Gamma$  possesses a potential barrier at the boundary  $\partial\Gamma$ , such that the charged particle  $q$  can not penetrate it and enter the region  $\Gamma$ , the value of  $\lambda(B) \in \mathbb{R}$ , defined by (8.61), remains constant. This entails that a suitable outside the region  $\Gamma$  measurement can certainly indicate the presence of the magnetic field inside  $\Gamma$ . So, as it was mentioned in [67], the constructed above current algebra representation completely describes our non-relativistic quasi-stationary system not giving rise to the Aharonov-Bohm [2] paradox. Moreover, the outside measurements results simply depend on the representation of the current algebra (8.52), which in turn depends on the history of the system and the topology of the space outside the barrier. As a related physical aspect of the explanation above it is necessary to stress that vanishing of the magnetic field outside the region  $\Gamma$ , possessing a nontrivial topology, does not imply the simultaneous vanishing of the corresponding magnetic potential outside the region  $\Gamma$ . Namely the latter in somewhat obscured form was used in the analysis of the current algebra representation, suitable for describing the complete set of physical observables both inside and outside the region  $\Gamma$ .

### 8.3. The Fock space embedding method, nonlinear dynamical systems and their complete linearization.

This final Section is devoted to an interesting application of the

Consider now the case, when the basic Fock space  $\Phi = \otimes_{j=1}^s \Phi^{(j)}$ , where  $\Phi^{(j)}$ ,  $j = \overline{1, s}$ , are Fock spaces corresponding to the different types of independent cyclic vectors  $|\Omega_j\rangle \in \Phi^{(j)}$ ,  $j = \overline{1, s}$ . This, in particular, means that the suitably constructed creation and annihilation operators  $a_j(x), a_k^+(y) : \Phi \longrightarrow \Phi$ ,  $j, k = \overline{1, s}$ , satisfy the following commutation relations:

$$(8.62) \quad \begin{aligned} [a_j(x), a_k(y)] &= 0, \\ [a_j(x), a_k^+(y)] &= \delta_{jk} \delta(x - y) \end{aligned}$$

for any  $x, y \in \mathbb{R}^m$ .

**Definition 8.14.** A vector  $|u\rangle \in \Phi$ ,  $x \in \mathbb{R}^m$ , is called coherent [63, 123] with respect to a mapping  $u \in L_2(\mathbb{R}^m; \mathbb{R}^s) := M$ , if it satisfies the eigenfunction condition

$$(8.63) \quad a_j(x)|u\rangle = u_j(x)|u\rangle$$

for each  $j = \overline{1, s}$  and all  $x \in \mathbb{R}^m$ .

It is easy to check that the coherent vectors  $|u\rangle \in \Phi$  exist. Really, the following vector expression

$$(8.64) \quad |u\rangle := \exp\{(u, a^+)\}|\Omega\rangle,$$

where the cyclic state  $|\Omega\rangle := \otimes_{j=1}^m |\Omega_j\rangle$  and  $(\cdot, \cdot)$  is the standard scalar product in the Hilbert space  $M$ , satisfies the defining condition (8.63), and moreover, its norm

$$(8.65) \quad \|u\|_\Phi := \langle u|u\rangle^{1/2} = \exp(\frac{1}{2}\|u\|^2) < \infty,$$

since  $u \in M$  and the norm  $\|u\| := (u, u)^{1/2}$  is bounded.

Take now any function  $u \in M := L_2(\mathbb{R}^m; \mathbb{R}^s)$  and observe that the Fock space embedding mapping

$$(8.66) \quad \phi : M \ni u \longrightarrow |u\rangle \in \Phi,$$

defined by means of the coherent vector expression (8.64) realizes a smooth isomorphism between Hilbert spaces  $M$  and  $\Phi$ . The inverse mapping  $\phi^{-1} : \Phi \longrightarrow M$  is given by the following exact expression:

$$(8.67) \quad u(x) = \langle \Omega | a(x) | u \rangle,$$

holding for almost all  $x \in \mathbb{R}^m$ . Owing to condition (8.65), one finds from (8.67) that, the corresponding function  $u \in M$ .

In the Hilbert space  $M$ , let now define a nonlinear dynamical system (which can, in general, be non-autonomous) in partial derivatives

$$(8.68) \quad du/dt = K[u],$$

where  $t \in \mathbb{R}_+$  is the corresponding evolution parameter,  $[u] := (t, x; u, u_x, u_{xx}, \dots, u_{rx})$ ,  $r \in \mathbb{Z}_+$ , and a mapping  $K : M \longrightarrow T(M)$  is Frechet smooth. Assume also that the Cauchy problem

$$(8.69) \quad u|_{t=+0} = u_0$$

is solvable for any  $u_0 \in M$  in an interval  $[0, T) \subset \mathbb{R}_+^1$  for some  $T > 0$ . Thereby, the smooth evolution mapping is defined

$$(8.70) \quad T_t : M \ni u_0 \longrightarrow u(t|u_0) \in M,$$

for all  $t \in [0, T)$ .

It is now natural to consider the following commuting diagram

$$(8.71) \quad \begin{array}{ccc} M & \xrightarrow{\phi} & \Phi \\ T_t \downarrow & & \downarrow \mathbb{T}_t \\ M & \xrightarrow{\phi} & \Phi, \end{array}$$

where the mapping  $\mathbb{T}_t : \Phi \longrightarrow \Phi$ ,  $t \in [0, T)$ , is defined from the conjugation relationship

$$(8.72) \quad \phi \circ T_t = \mathbb{T}_t \circ \phi$$

Now take coherent vector  $|u_0\rangle \in \Phi$ , corresponding to  $u_0 \in M$ , and construct the vector

$$(8.73) \quad |u\rangle := \mathbb{T}_t |u_0\rangle$$

for all  $t \in [0, T)$ . Since vector (8.73) is, by construction, coherent, that is

$$(8.74) \quad a_j(x)|u\rangle := u_j(x, t|u_0)|u\rangle$$

for each  $j = \overline{1, s}$ ,  $t \in [0, T)$  and almost all  $x \in \mathbb{R}^m$ , owing to the smoothness of the mapping  $\xi : M \rightarrow \Phi$  with respect to the corresponding norms in the Hilbert spaces  $M$  and  $\Phi$ , we derive that coherent vector (8.73) is differentiable with respect to the evolution parameter  $t \in [0, T)$ . Thus, one can easily find [93, 92, 123, 124] that

$$(8.75) \quad \frac{d}{dt}|u\rangle = \hat{K}[a^+, a]|u\rangle,$$

where

$$(8.76) \quad |u\rangle|_{t=+0} = |u_0\rangle$$

and a mapping  $\hat{K}[a^+, a] : \Phi \rightarrow \Phi$  is defined by the exact analytical expression

$$(8.77) \quad \hat{K}[a^+, a] := (a^+, K[a]).$$

As a result of the consideration above we obtain the following theorem.

**Theorem 8.15.** *Any smooth nonlinear dynamical system (8.68) in Hilbert space  $M := L_2(\mathbb{R}^m; \mathbb{R}^s)$  is representable by means of the Fock space embedding isomorphism  $\phi : M \rightarrow \Phi$  in the completely linear form (8.75).*

We now make some comments concerning the solution to the linear equation (8.75) under the Cauchy condition (8.76). Since any vector  $|u\rangle \in \Phi$  allows the series representation

$$(8.78) \quad |u\rangle = \bigoplus_{n=\sum_{j=1}^s n_j \in \mathbb{Z}_+} \frac{1}{(n_1!n_2!\dots n_s!)^{1/2}} \int_{(\mathbb{R}^m)^n} f_{n_1 n_2 \dots n_s}^{(n)}(x_1^{(1)}, x_2^{(1)}, \dots, x_{n_1}^{(1)}; x_1^{(2)}, x_2^{(2)}, \dots, x_{n_2}^{(2)}; \dots; x_1^{(s)}, x_2^{(s)}, \dots, x_{n_s}^{(s)}) \prod_{j=1}^s \left( \prod_{k=1}^{n_j} dx_k^{(j)} a_j^+(x_k^{(j)}) \right) |\Omega\rangle,$$

where for any  $n = \sum_{j=1}^s n_j \in \mathbb{Z}_+$  functions

$$(8.79) \quad f_{n_1 n_2 \dots n_s}^{(n)} \in \bigotimes_{j=1}^s L_{2,s}((\mathbb{R}^m)^{n_j}; \mathbb{C}) \simeq L_{2,s}(\mathbb{R}^{mn_1} \times \mathbb{R}^{mn_2} \times \dots \mathbb{R}^{mn_s}; \mathbb{C}),$$

and the norm

$$(8.80) \quad \|u\|_{\Phi}^2 = \sum_{n=\sum_{j=1}^s n_j \in \mathbb{Z}_+} \|f_{n_1 n_2 \dots n_s}^{(n)}\|_2^2 = \exp(\|u\|^2).$$

By substituting (8.78) into equation (8.75), reduces (8.75) to an infinite recurrent set of linear evolution equations in partial derivatives on coefficient functions (8.79). The latter can often be solved [92, 123, 124] step by step analytically in exact form, thereby, making it possible to obtain, owing to representation (8.67), the exact solution  $u \in M$  to the Cauchy problem (8.69) for our nonlinear dynamical system in partial derivatives (8.68).

*Remark 8.16.* Concerning some applications of nonlinear dynamical systems like (8.66) in mathematical physics problems, it is very important to construct their so called conservation laws or smooth invariant functionals  $\gamma : M \rightarrow \mathbb{R}$  on  $M$ . Making use of the quantum mathematics technique described above one can suggest an effective algorithm for constructing these conservation laws in exact form.

Indeed, consider a vector  $|\gamma\rangle \in \Phi$ , satisfying the linear equation:

$$(8.81) \quad \frac{\partial}{\partial t}|\gamma\rangle + \hat{K}^*[a^+, a]|\gamma\rangle = 0.$$

Then, the following proposition [92, 123, 124] holds.

**Proposition 8.17.** *The functional*

$$(8.82) \quad \gamma := \langle u | \gamma \rangle$$

*is a conservation law for dynamical system (8.66), that is*

$$(8.83) \quad d\gamma/dt|_K = 0$$

*along any orbit of the evolution mapping (8.70).*

**8.4. Comments.** Within the scope of this Section we have described general aspects of the current algebra representations approach to describing quantum dynamical systems in Fock type spaces. The problem of constructing physically realizable quantum states was analyzed in much detail using the Gelfand-Vilenkin representation theory [61, 22] of infinite dimensional groups and the Goldin-Menikoff-Sharp theory [66, 65, 67] of generating Bogolubov type functionals, classifying these representations. The related problem of constructing Fock type space representations and retrieving their creation-annihilation generating structure still needs a deeper investigation within the approach devised. Here we mention only that some aspects of this problem within the so-called Poissonian White noise analysis were studied in a series of works [14, 13, 3, 89, 102], based on some generalizations of the Delsarte type characters technique. Based on the quantum current Lie algebra description of a bounded non-relativistic quantum system under an external electromagnetic field within the *minimal interaction* principle, the magnetic Aharonov-Bohm effect has been interpreted. As it was mentioned in [67], the suitably constructed current algebra representation completely describes our non-relativistic quasi-stationary system not giving rise to the Aharonov-Bohm [2] paradox.

We have also presented main mathematical preliminaries and properties of the related quantum mathematics techniques suitable for analytical studying of the important linearization problem for a wide class of nonlinear dynamical systems in partial derivatives in Hilbert spaces. Concerning this direction it is worthy to mention the related results also obtained in [91, 92, 93, 125, 126, 138], devoted to the application of the Fock space embedding method to studying solutions to a wide nonlinear dynamical systems and to constructing quantum computing algorithms.

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